Lecture Notes 3
Two Random Variables

- Joint, Marginal, and Conditional PMFs
- Joint, Marginal, and Conditional CDFs, PDFs
- One Discrete and one Continuous Random Variables
- Signal Detection: MAP Rule
- Functions of Two Random Variables

Joint, Marginal, and Conditional PMFs

- Let X and Y be discrete random variables on the same probability space
- They are completely specified by their *joint pmf*:

$$p_{X,Y}(x,y) = \mathrm{P}\{X = x, Y = y\}\,, \quad x \in \mathcal{X},\, y \in \mathcal{Y}$$

By axioms of probability, $\displaystyle\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p_{X,Y}(x,y) = 1$

- Example: Consider the pmf $p_{X,Y}(x,y)$ described by the following table

		x = 0	x = 1	x = 2.5
y	-3	0	$\frac{1}{4}$	$\frac{1}{8}$
	-1	$\frac{1}{8}$	0	$\frac{1}{4}$
	2	$\frac{1}{8}$	$\frac{1}{8}$	0

- To find $p_X(x)$, the *marginal pmf* of X, we use the law of total probability

$$p_X(x) = \sum_{y \in \mathcal{Y}} p(x, y)\,, \quad x \in \mathcal{X}$$

- The *conditional pmf* of X given $Y = y$ is defined as

$$p_{X|Y}(x|y) \;=\; \frac{p_{X,Y}(x,y)}{p_Y(y)}\,, \quad p_Y(y) \neq 0,\, x \in \mathcal{X}$$

  Check that if $p_Y(y) \neq 0$ then $p_{X|Y}(x|y)$ is a pmf for X The (elementary) conditional probability of an event $X \in A$ given $Y = y$ is

$$\mathrm{P}(X \in A \mid Y = y) = \sum_{x \in A} p_{X|Y}(x|y)$$

- *Chain rule*: $p_{X,Y}(x,y) = p_X(x)p_{Y|X}(y|x) = p_Y(y)p_{X|Y}(x|y)$
- X and Y are said to be *independent* if for every $(x, y) \in \mathcal{X} \times \mathcal{Y}$,

$$p_{X,Y}(x,y) = p_X(x)p_Y(y)\,,$$

  which is equivalent to

$$p_{X|Y}(x|y) \;=\; p_X(x)\,, \quad p_Y(y) \neq 0,\, x \in \mathcal{X}$$

Bayes Rule for PMFs

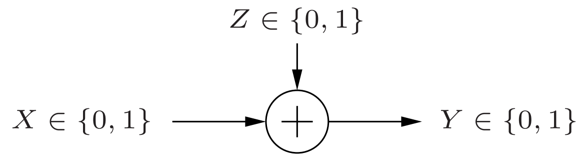
Given $p_X(x)$ and $p_{Y|X}(y|x)$ for every $(x, y) \in \mathcal{X} \times \mathcal{Y}$, we can find $p_{X|Y}(x|y)$:

$$\begin{aligned} p_{X|Y}(x|y) &= \frac{p_{X,Y}(x,y)}{p_Y(y)} \\ &= \frac{p_X(x)p_{Y|X}(y|x)}{p_Y(y)} \\ &= \frac{p_{Y|X}(y|x)}{\sum_{x' \in \mathcal{X}} p_{X,Y}(x',y)} p_X(x) \\ &= \frac{p_{Y|X}(y|x)}{\sum_{x' \in \mathcal{X}} p_{Y|X}(y|x')p_X(x')}\, p_X(x) \end{aligned}$$

The final formula is entirely in terms of the known quantities $p_X(x)$ and $p_{Y|X}(y|x)$

Example: Binary Symmetric Channel

Consider the following binary communication channel



The bit sent is $X \sim \mathrm{Bern}(p)$, $0 \le p \le 1$, the noise is $Z \sim \mathrm{Bern}(\epsilon)$, $0 \le \epsilon \le 0.5$, the bit received is $Y = (X + Z) \bmod 2 = X \oplus Z$, and X and Z are independent

Find

1. $p_{X|Y}(x|y)$
2. $p_Y(y)$
3. $\mathrm{P}\{X \neq Y\}$, the probability of error

1. To find $p_{X|Y}(x|y)$ we use Bayes rule

$$p_{X|Y}(x|y) = \frac{p_{Y|X}(y|x)}{\sum\limits_{x' \in \mathcal{X}} p_{Y|X}(y|x') p_X(x')} \, p_X(x)$$

We know $p_X(x)$, but we need to find $p_{Y|X}(y|x)$:

$$\begin{aligned}
p_{Y|X}(y|x) &= \mathrm{P}\{Y = y \,|\, X = x\} = \mathrm{P}\{X \oplus Z = y \,|\, X = x\} \\
&= \mathrm{P}\{x \oplus Z = y \,|\, X = x\} = \mathrm{P}\{Z = y \oplus x \,|\, X = x\} \\
&= \mathrm{P}\{Z = y \oplus x\} \quad \text{since } Z \text{ and } X \text{ are independent} \\
&= p_Z(y \oplus x)
\end{aligned}$$

Therefore

$$\begin{aligned}
p_{Y|X}(0\,|\,0) &= p_Z(0 \oplus 0) = p_Z(0) = 1 - \epsilon \\
p_{Y|X}(0\,|\,1) &= p_Z(0 \oplus 1) = p_Z(1) = \epsilon \\
p_{Y|X}(1\,|\,0) &= p_Z(1 \oplus 0) = p_Z(1) = \epsilon \\
p_{Y|X}(1\,|\,1) &= p_Z(1 \oplus 1) = p_Z(0) = 1 - \epsilon
\end{aligned}$$

Plugging into the Bayes rule equation, we obtain

$$p_{X|Y}(0|0) = \frac{p_{Y|X}(0|0)}{p_{Y|X}(0|0)p_X(0) + p_{Y|X}(0|1)p_X(1)} p_X(0) = \frac{(1-\epsilon)(1-p)}{(1-\epsilon)(1-p)+\epsilon p}$$

$$p_{X|Y}(1|0) = 1 - p_{X|Y}(0|0) = \frac{\epsilon p}{(1-\epsilon)(1-p)+\epsilon p}$$

$$p_{X|Y}(0|1) = \frac{p_{Y|X}(1|0)}{p_{Y|X}(1|0)p_X(0) + p_{Y|X}(1|1)p_X(1)} p_X(0) = \frac{\epsilon(1-p)}{(1-\epsilon)p+\epsilon(1-p)}$$

$$p_{X|Y}(1|1) = 1 - p_{X|Y}(0|1) = \frac{(1-\epsilon)p}{(1-\epsilon)p+\epsilon(1-p)}$$

2. We already found $p_Y(y)$ as

$$\begin{aligned} p_Y(y) &= p_{Y|X}(y|0)p_X(0) + p_{Y|X}(y|1)p_X(1) \\ &= \begin{cases} (1-\epsilon)(1-p)+\epsilon p & \text{for } y = 0 \\ \epsilon(1-p)+(1-\epsilon)p & \text{for } y = 1 \end{cases} \end{aligned}$$

3. Now to find the probability of error $\mathrm{P}\{X \neq Y\}$, consider

$$\begin{aligned} \mathrm{P}\{X \neq Y\} &= p_{X,Y}(0,1) + p_{X,Y}(1,0) \\ &= p_{Y|X}(1|0)p_X(0) + p_{Y|X}(0|1)p_X(1) \\ &= \epsilon(1-p) + \epsilon p \;=\; \epsilon \end{aligned}$$

An interesting special case is $\epsilon = \frac{1}{2}$. Here, $\mathrm{P}\{X \neq Y\} = \frac{1}{2}$, which is the worst possible (no information is sent), and

$$p_Y(0) = \tfrac{1}{2}p + \tfrac{1}{2}(1-p) = \tfrac{1}{2} = p_Y(1)$$

Therefore $Y \sim \mathrm{Bern}(\frac{1}{2})$, independent of the value of p !

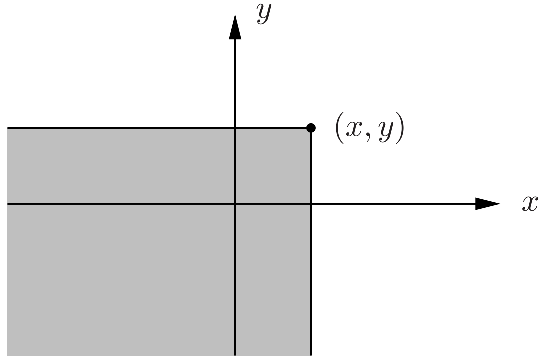
In this case, the bit sent X and the bit received Y are independent (check this)

Joint and Marginal CDF and PDF

- Any two random variables are specified by their *joint cdf*

$$F_{X,Y}(x,y) = \mathrm{P}\{X \le x,\, Y \le y\}\,, \quad x,y \in \mathbf{R}$$

$F_{X,Y}(x,y)$ is the probability of the shaded region of $\mathbf{R}^2$

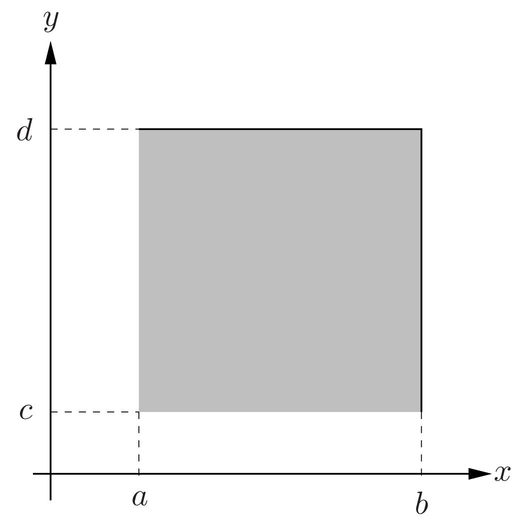


- Properties of the cdf:
  - $F_{X,Y}(x,y) \ge 0$
  - If $x_1 \le x_2$ and $y_1 \le y_2$ then $F_{X,Y}(x_1,y_1) \le F_{X,Y}(x_2,y_2)$
  - $\lim_{x,y\to\infty} F_{X,Y}(x,y) = 1$
  - $\lim_{y\to-\infty} F_{X,Y}(x,y) = 0$ and $\lim_{x\to-\infty} F_{X,Y}(x,y) = 0$
  - $\lim_{y\to\infty} F_{X,Y}(x,y) = F_X(x)$ and $\lim_{x\to\infty} F_{X,Y}(x,y) = F_Y(y)$

    This is often abbreviated to $F_{X,Y}(x,\infty) = F_X(x)$, $F_{X,Y}(\infty,y) = F_Y(y)$.

    $F_X(x)$ and $F_Y(y)$ are called the *marginal cdfs* of X and Y

◦ The probability of any rectangular set can be determined from the joint cdf



For example,

$$\mathrm{P}\{a < X \le b,\, c < Y \le d\} = F(b,d) - F(a,d) - F(b,c) + F(a,c)$$

- X and Y are *independent* if for every x and y

$$F_{X,Y}(x,y) = F_X(x)F_Y(y)$$

- X and Y are jointly *continuous* random variables if their joint cdf is continuous in both x and y

  In this case, we can define their *joint pdf*, provided that it exists, as the function $f_{X,Y}(x,y)$ such that

$$F_{X,Y}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(u,v)\,du\,dv\,, \quad x,y \in \mathbf{R}$$

- If $F_{X,Y}(x,y)$ is differentiable in x and y, then

$$f_{X,Y}(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y} = \lim_{\Delta x, \Delta y \to 0} \frac{\mathrm{P}\{x < X \le x + \Delta x,\ y < Y \le y + \Delta y\}}{\Delta x \Delta y}$$

- Properties of $f_{X,Y}(x,y)$:
  - $f_{X,Y}(x,y) \geq 0$
  - $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$
  - The probability of any set $A \subset \mathbf{R}$ can be calculated by integrating the joint pdf over A:

$$\mathrm{P}\{(X,Y) \in A\} = \int_{(x,y)\in A} f_{X,Y}(x,y)\,dx\,dy$$

- The *marginal pdf* of X can be obtained from the joint pdf via the law of total probability:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

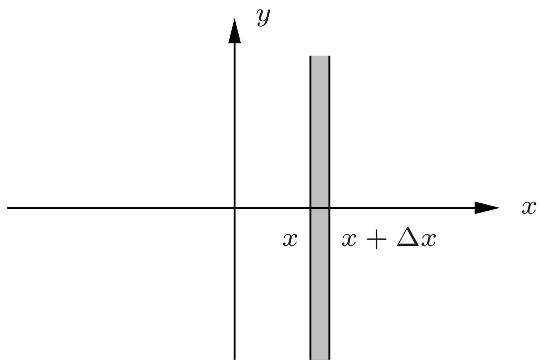
To see this,

$$\begin{aligned} F_X(x) &= P\{X \leq x\} = P\{X \leq x, Y \leq \infty\} \\ & \infty \\ &= \int_{-\infty}^{x}\int_{-\infty}^{\infty} f_{X,Y}(x',y)dx'dy \end{aligned}$$

and hence differentiating yields

$$f_X(x) = \frac{d}{dx}\left(\int_{-\infty}^{x}\int_{-\infty}^{\infty} f_{X,Y}(x',y)dx'dy\right) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$$

Alternatively, consider the figure on the next page



$$
\begin{aligned}
f_X(x) &= \lim_{\Delta x \to 0} \frac{\mathrm{P}\{x < X \le x + \Delta x\}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \lim_{\Delta y \to 0} \sum_{n=-\infty}^{\infty} \mathrm{P}\{x < X \le x + \Delta x,\, n\Delta y < Y \le (n+1)\Delta y\} \\
&= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy\, \Delta x = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy
\end{aligned}
$$

- X and Y are independent iff $f_{X,Y}(x,y) = f_X(x) f_Y(y)$ for every x, y

Example

- Let $(X,Y) \sim f(x,y)$, where

$$
f(x,y) = \begin{cases} c & x \ge 0,\, y \ge 0,\, x + y \le 1 \\ 0 & \text{otherwise} \end{cases}
$$

1. Find c
2. Find $f_Y(y)$
3. Are X and Y independent?
4. Find $\mathrm{P}\{X \ge \frac{1}{2} Y\}$

Solution:

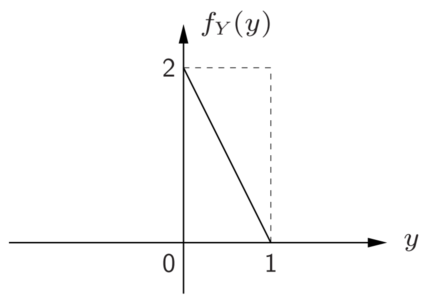
1. To find c, note that

$$
1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\, dx\, dy \; = \int_0^1 \int_0^{1-y} c\, dx\, dy = c \int_0^1 (1-y)\, dy = \tfrac{1}{2} c\,,
$$

hence $c = 2$

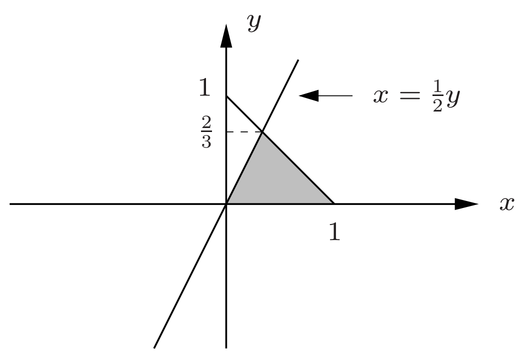
2. To find $f_Y(y)$, we use the law of total probability

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$
$$= \begin{cases} \int_0^{(1-y)} 2\,dx & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 2(1-y) & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$



3. X and Y are independent if $f_{X,Y}(x,y) = f_X(x) f_Y(y)$ for every x, y
   But $f_{X,Y}(0,1) = 2$ and $f_X(0) f_Y(1) = 0$, so X and Y are *not* independent
   Another example: $f_{X,Y}(\frac{1}{2},\frac{1}{2}) = 2 \ne 1 \cdot 1 = f_X(\frac{1}{2}) f_Y(\frac{1}{2})$

4. To find the probability of the set $\{X \ge \frac{1}{2}Y\}$ we first sketch the set



From the figure we find that

$$\mathrm{P}\left\{X \ge \tfrac{1}{2}Y\right\} = \int_{\left\{(x,y):x \ge \frac{1}{2}y\right\}} f_{X,Y}(x,y)\,dx\,dy$$
$$= \int_0^{\frac{2}{3}} \int_{\frac{y}{2}}^{(1-y)} 2\,dx\,dy = \tfrac{2}{3}$$

Conditional CDF and PDF

- Let X and Y be continuous random variables with joint pdf $f_{X,Y}(x, y)$. We wish to define $F_{Y|X}(y \,|\, X = x) = \mathrm{P}\{Y \le y \,|\, X = x\}$
- We cannot define the above conditional probability as

$$\frac{\mathrm{P}\{Y \le y,\, X = x\}}{\mathrm{P}\{X = x\}}$$

because both numerator and denominator are equal to zero. Instead, we define conditional probability for continuous random variables as a limit

$$\begin{aligned} F_{Y|X}(y|x) &= \lim_{\Delta x \to 0} \mathrm{P}\{Y \le y \,|\, x < X \le x + \Delta x\} \\ &= \lim_{\Delta x \to 0} \frac{\mathrm{P}\{Y \le y,\, x < X \le x + \Delta x\}}{\mathrm{P}\{x < X \le x + \Delta x\}} \\ &= \lim_{\Delta x \to 0} \frac{\int_{-\infty}^{y} f_{X,Y}(x, u)\, du\, \Delta x}{f_X(x)\Delta x} = \int_{-\infty}^{y} \frac{f_{X,Y}(x, u)}{f_X(x)}\, du \end{aligned}$$

- We then define the conditional pdf in the usual way as

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x, y)}{f_X(x)} \qquad \text{if } f_X(x) \ne 0$$

- Thus

$$F_{Y|X}(y|x) = \int_{-\infty}^{y} f_{Y|X}(u|x)\, du$$

which shows that $f_{Y|X}(y|x)$ is a pdf for Y given $X = x$, i.e.,

$$Y \mid \{X = x\} \sim f_{Y|X}(y|x)$$

Note: Conditional distributions are in fact not defined in this way in advanced probability, in particular they are not defined using limits. Like Dirac delta functions, they are defined by their behavior inside integrals — suitable integrals of conditional pdfs yield elementary conditional probabilities. See Section 3.7 of G&D.

- Example: Let $f(x, y)$ be defined as

$$f(x, y) = \begin{cases} 2 & x \ge 0,\, y \ge 0,\, x + y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $f_{X|Y}(x|y)$

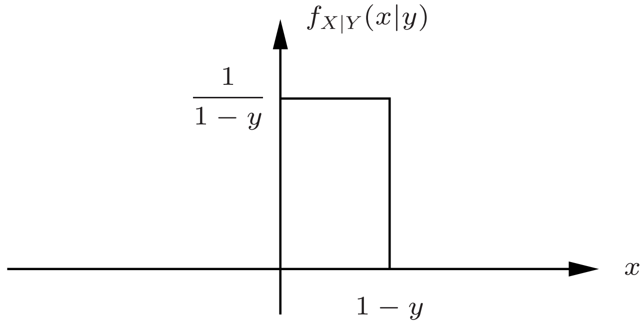
Solution: We already know that

$$f_Y(y) = \begin{cases} 2(1-y) & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Therefore

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \dfrac{1}{1-y} & 0 \le y < 1,\, 0 \le x \le 1-y \\ 0 & \text{otherwise} \end{cases}$$

In other words, $X \mid \{Y = y\} \sim \mathrm{U}[\,0, 1-y\,]$



- Bayes rule for densities: Given $f_X(x)$ and $f_{Y|X}(y|x)$, we can find

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f_{Y|X}(y|x)}{f_Y(y)} f_X(x) \\ &= \frac{f_{Y|X}(y|x)}{\int_{-\infty}^{\infty} f_{X,Y}(u,y)\,du} f_X(x) \\ &= \frac{f_{Y|X}(y|x)}{\int_{-\infty}^{\infty} f_X(u) f_{Y|X}(y|u)\,du} f_X(x) \end{aligned}$$

- Example: Let $\Lambda \sim \mathrm{U}[0,1]$, and let the conditional pdf of X given $\Lambda = \lambda$ be

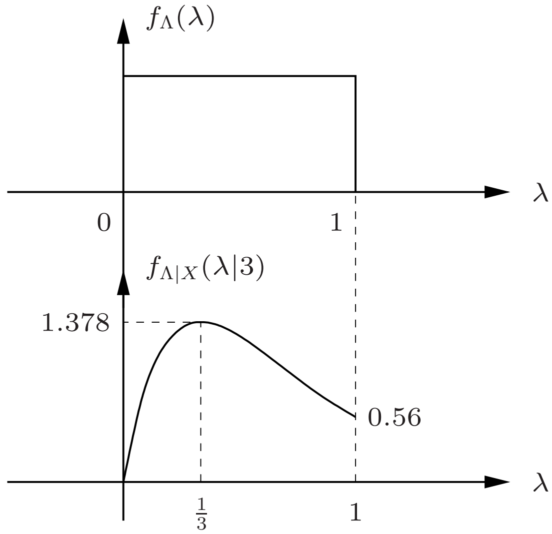
$$f_{X|\Lambda}(x|\lambda) = \lambda e^{-\lambda x}, \quad 0 < \lambda \le 1,$$

i.e., $X \mid \{\Lambda = \lambda\} \sim \mathrm{Exp}(\lambda)$

Given $X = 3$, find $f_{\Lambda|X}(\lambda|3)$

Solution: Use Bayes rule

$$f_{\Lambda|X}(\lambda|3) = \frac{f_{X|\Lambda}(3|\lambda) f_\Lambda(\lambda)}{\int_0^1 f_\Lambda(u) f_{X|\Lambda}(3|u)\, du} = \begin{cases} \frac{\lambda e^{-3\lambda}}{\frac{1}{9}(1-4e^{-3})} & 0 < \lambda \le 1 \\ 0 & \text{otherwise} \end{cases}$$



Mixed Random Variables

- Let Θ be a discrete random variable with pmf $p_\Theta(\theta)$
- For each $\Theta = \theta$ with $p_\Theta(\theta) \neq 0$, let Y be a continuous random variable, i.e., $F_{Y|\Theta}(y|\theta)$ is continuous for all θ. We define $f_{Y|\Theta}(y|\theta)$ in the usual way
- The conditional pmf of Θ given y can be defined as a limit

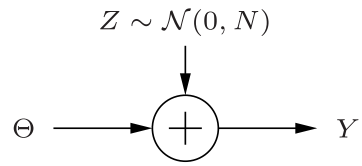
$$\begin{aligned} p_{\Theta|Y}(\theta \,|\, y) &= \lim_{\Delta y \to 0} \frac{\mathrm{P}\{\Theta = \theta,\, y < Y \le y + \Delta y\}}{\mathrm{P}\{y < Y \le y + \Delta y\}} \\ &= \lim_{\Delta y \to 0} \frac{p_\Theta(\theta) f_{Y|\Theta}(y|\theta) \Delta y}{f_Y(y) \Delta y} = \frac{f_{Y|\Theta}(y|\theta)}{f_Y(y)} p_\Theta(\theta) \end{aligned}$$

- So we obtain yet another version of Bayes rule: Given $p_\Theta(\theta)$ and $f_{Y|\Theta}(y|\theta)$, then

$$p_{\Theta|Y}(\theta \,|\, y) = \frac{f_{Y|\Theta}(y|\theta)}{\sum_{\theta'} p_\Theta(\theta') f_{Y|\Theta}(y|\theta')} p_\Theta(\theta)$$

- Example: *Additive Gaussian Noise Channel*

  Consider the following communication channel:



  The signal transmitted is a binary random variable Θ:

$$\Theta = \begin{cases} +1 & \text{with probability } p \\ -1 & \text{with probability } 1-p \end{cases}$$

  The received signal, also called the *observation*, is $Y = \Theta + Z$, where Θ and Z are independent

  Given $Y = y$ is received (observed), find $p_{\Theta|Y}(\theta|y)$, the a posteriori pmf of Θ

Solution: We use Bayes rule

$$p_{\Theta|Y}(\theta|y) = \frac{f_{Y|\Theta}(y|\theta)}{\sum_{\theta'} p_\Theta(\theta') f_{Y|\Theta}(y|\theta')} p_\Theta(\theta)$$

We are given $p_\Theta(\theta)$:

$$p_\Theta(+1) \, = \, p \quad \text{and} \quad p_\Theta(-1) \, = \, 1-p$$

and $f_{Y|\Theta}(y|\theta) = f_Z(y-\theta)$:

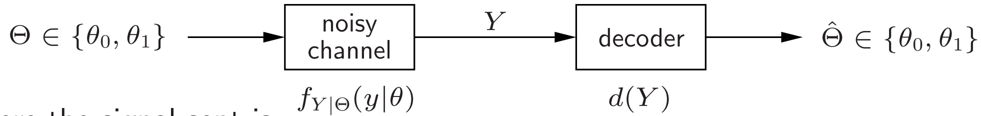
$$Y \,|\, \{\Theta = +1\} \sim \mathcal{N}(+1, N) \quad \text{and} \quad Y \,|\, \{\Theta = -1\} \sim \mathcal{N}(-1, N)$$

Therefore

$$p_{\Theta|Y}(1|y) \, = \, \frac{\frac{p}{\sqrt{2\pi N}} e^{-\frac{(y-1)^2}{2N}}}{\frac{p}{\sqrt{2\pi N}} e^{-\frac{(y-1)^2}{2N}} + \frac{(1-p)}{\sqrt{2\pi N}} e^{-\frac{(y+1)^2}{2N}}} \, = \, \frac{p e^{\frac{y}{N}}}{p e^{\frac{y}{N}} + (1-p) e^{-\frac{y}{N}}}$$

Signal Detection

- Consider the following general digital communication system



where the signal sent is

$$\Theta = \begin{cases} \theta_0 & \text{with probability } p \\ \theta_1 & \text{with probability } 1-p \end{cases}$$

and the observation (received signal) is

$$Y \,|\, \{\Theta = \theta\} \sim f_{Y|\Theta}(y \,|\, \theta)\,, \quad \theta \in \{\theta_0, \theta_1\}$$

- We wish to find an optimal decoder (or optimal receiver), $d(Y)$, that minimizes the *probability of error*:

$$\begin{aligned} P_e &\stackrel{\triangle}{=} \mathrm{P}\{\hat{\Theta} \neq \Theta\} = \mathrm{P}\{\Theta = \theta_0,\ \hat{\Theta} = \theta_1\} + \mathrm{P}\{\Theta = \theta_1,\ \hat{\Theta} = \theta_0\} \\ &= \mathrm{P}\{\Theta = \theta_0\}\mathrm{P}\{\hat{\Theta} = \theta_1 \,|\, \Theta = \theta_0\} + \mathrm{P}\{\Theta = \theta_1\}\mathrm{P}\{\hat{\Theta} = \theta_0 \,|\, \Theta = \theta_1\} \end{aligned}$$

- We define the *maximum a posteriori probability* (MAP) decoder as

$$d(y) \ = \begin{cases} \theta_0 & \text{if } p_{\Theta|Y}(\theta_0|y) > p_{\Theta|Y}(\theta_1|y) \\ \theta_1 & \text{otherwise} \end{cases}$$

- The MAP decoding rule minimizes P_e, since

$$\begin{aligned} P_e &= 1 - \mathrm{P}\{d(Y) = \Theta\} \\ &= 1 - \int_{-\infty}^{\infty} f_Y(y)\mathrm{P}\{d(y) = \Theta \,|\, Y = y\}\, dy \end{aligned}$$

and the integral is maximized when we pick the largest $\mathrm{P}\{d(y) = \Theta \,|\, Y = y\}$ for each y, which is precisely the MAP decoder

- If $p = \frac{1}{2}$, i.e., equally likely signals, using Bayes rule, the MAP decoder reduces to the *maximum likelihood* (ML) decoder

$$d(y) \ = \begin{cases} \theta_0 & \text{if } f_{Y|\Theta}(y|\theta_0) > f_{Y|\Theta}(y|\theta_1) \\ \theta_1 & \text{otherwise} \end{cases}$$

Additive Gaussian Noise Channel

- Consider the additive Gaussian noise channel with signal

$$\Theta = \begin{cases} +\sqrt{P} & \text{with probability } \frac{1}{2} \\ -\sqrt{P} & \text{with probability } \frac{1}{2} \end{cases}$$

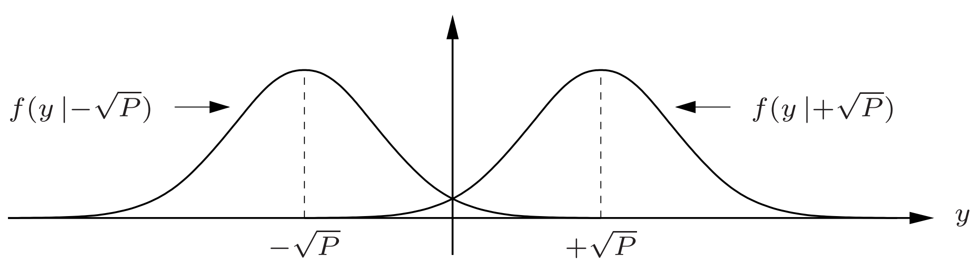
noise $Z \sim \mathcal{N}(0, N)$ (Θ and Z are independent), and output $Y = \Theta + Z$

- The MAP decoder is

$$d(y) = \begin{cases} +\sqrt{P} & \text{if } \dfrac{\mathrm{P}\{\Theta = +\sqrt{P} \mid Y = y\}}{\mathrm{P}\{\Theta = -\sqrt{P} \mid Y = y\}} > 1 \\ -\sqrt{P} & \text{otherwise} \end{cases}$$

Since the two signals are equally likely, the MAP decoding rule reduces to the ML decoding rule

$$d(y) = \begin{cases} +\sqrt{P} & \text{if } \dfrac{f_{Y|\Theta}(y \mid +\sqrt{P})}{f_{Y|\Theta}(y \mid -\sqrt{P})} > 1 \\ -\sqrt{P} & \text{otherwise} \end{cases}$$



- From the figure above and using the Gaussian pdf, the MAP decoder reduces to the *minimum distance decoder*

$$d(y) = \begin{cases} +\sqrt{P} & (y - \sqrt{P})^2 < (y - (-\sqrt{P}))^2 \\ -\sqrt{P} & \text{otherwise} \end{cases}$$

which simplifies to

$$d(y) = \begin{cases} +\sqrt{P} & y > 0 \\ -\sqrt{P} & y < 0 \end{cases}$$

Note: The decision when $y = 0$ is arbitrary

- Now to find the *minimum* probability of error, consider

$$\begin{aligned}
P_e &= \mathrm{P}\{d(Y) \neq \Theta\} \\
&= \mathrm{P}\{\Theta = \sqrt{P}\}\mathrm{P}\{d(Y) = -\sqrt{P} \,|\, \Theta = \sqrt{P}\} + \\
&\qquad \mathrm{P}\{\Theta = -\sqrt{P}\}\mathrm{P}\{d(Y) = \sqrt{P} \,|\, \Theta = -\sqrt{P}\} \\
&= \tfrac{1}{2}\mathrm{P}\{Y \leq 0 \,|\, \Theta = \sqrt{P}\} + \tfrac{1}{2}\mathrm{P}\{Y > 0 \,|\, \Theta = -\sqrt{P}\} \\
&= \tfrac{1}{2}\mathrm{P}\{Z \leq -\sqrt{P}\} + \tfrac{1}{2}\mathrm{P}\{Z > \sqrt{P}\} \\
&= Q\left(\sqrt{\frac{P}{N}}\right) = Q\left(\sqrt{\mathrm{SNR}}\right)
\end{aligned}$$

The probability of error is a decreasing function of P/N, the *signal-to-noise ratio* (SNR)

Functions of Two Random Variables

- Let $(X, Y) \sim f(x, y)$ and let $g(x, y)$ be a differentiable function. To find the pdf of $Z = g(X, Y)$, we first find the inverse image of $\{z < Z \leq z + \Delta z\}$ then find its probability expressed as a function of z and Δz
- Example: Let X and Y be independent r.v.s, with $X \sim f_X(x)$, and $Y \sim f_Y(y)$. Find the pdf of $Z = X + Y$
- Solution 1: Using approximation and limit arguments

$$\begin{aligned}
f_Z(z)\Delta z &\approx \mathrm{P}\{z < Z \leq z + \Delta z\} \\
&= \int_{-\infty}^{\infty} \mathrm{P}\{z < X + Y \leq z + \Delta z \,|\, X = x\} f_X(x)\, dx \\
&= \int_{-\infty}^{\infty} \mathrm{P}\{z < x + Y < z + \Delta z \,|\, X = x\} f_X(x)\, dx \\
&= \int_{-\infty}^{\infty} \mathrm{P}\{z - x < Y < z - x + \Delta z\} f_X(x)\, dx \\
&\approx \int_{-\infty}^{\infty} f_Y(z - x)\Delta z f_X(x)\, dx
\end{aligned}$$

Letting $\Delta z \to 0$,

$$f_Z(z) \,=\, \int_{-\infty}^{\infty} f_Y(z-x) f_X(x)\, dx\,,$$

which is the *convolution* of $f_X(x)$ and $f_Y(y)$

For example, if $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ and $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$ are independent, then it can be shown that

$$Z = X + Y \sim \mathcal{N}(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$$

- Solution 2: Direct derived distribution:

Independence of X and Y implies that the joint pdf is $f_{X,Y}(x,y) = f_X(x) f_Y(y)$. To find pdf f_Z, first find the cdf $F_Z(z)$ and then differentiate.

$$\begin{aligned}
F_Z(z) &= \Pr(Z \le z) = \Pr(X + Y \le z) \\
&= \int_{x,y:x+y\le z} f_{X,Y}(x,y)dxdy \\
&= \int_{x,y:x+y\le z} f_X(x) f_Y(y) dxdy = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{z-y} f_X(x)dx \right) f_Y(y)dy
\end{aligned}$$

The pdf is then found by differentiation using the rule for differentiating integrals

with respect to variables which appear only in the limits:

$$\begin{aligned}
f_Z(z) &= \frac{dF_Z(z)}{dz} = \frac{d}{dz} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{z-y} f_X(x)dx \right) f_Y(y)dy \\
&= \int_{-\infty}^{\infty} \left(\frac{d}{dz} \int_{-\infty}^{z-y} f_X(x)dx \right) f_Y(y)dy \\
&= \int_{-\infty}^{\infty} f_X(z-y) f_Y(y)dy
\end{aligned}$$

- The convolution result also holds for the sum of two independent discrete random variables (replacing pdfs with pmfs and integrals with sums)

  For example, if $X \sim \mathrm{Poisson}(\lambda_1)$ and $Y \sim \mathrm{Poisson}(\lambda_2)$ are independent, then $Z = X + Y \sim \mathrm{Poisson}(\lambda_1) * \mathrm{Poisson}(\lambda_2) = \mathrm{Poisson}(\lambda_1 + \lambda_2)$

- The property that the sum of two independent r.v.s with the same distribution has the same distribution, which is obeyed by Gaussian and Poisson r.v.s, is referred to as *infinite divisibility*

  E.g., a Poisson r.v. with parameter λ can be written as the sum of *any* number of independent $\mathrm{Poisson}(\lambda_i)$ r.v.s, so long as $\sum_i \lambda_i = \lambda$

- Example: *Minimum and Maximum of Independent Random Variables*

  Let $X \sim f_X(x)$ and $Y \sim f_Y(y)$ be independent. Define

$$U = \max\{X, Y\} \quad \text{and} \quad V = \min\{X, Y\}$$

  Find the pdfs of U and V

  Solution: To find the pdf of U, we first find its cdf:

$$F_U(u) = \mathrm{P}\{U \le u\} = \mathrm{P}\{X \le u, Y \le u\} = F_X(u)F_Y(u)$$

  Using the product rule for derivatives,

$$f_U(u) = f_X(u)F_Y(u) + f_Y(u)F_X(u)$$

  Now to find the pdf of V,

$$\mathrm{P}\{V > v\} = \mathrm{P}\{X > v, Y > v\} \Rightarrow 1 - F_V(v) = (1 - F_X(v))(1 - F_Y(v))$$

  Thus

$$f_V(v) = f_X(v) + f_Y(v) - f_X(v)F_Y(v) - f_Y(v)F_X(v)$$