

Elementary Properties of Probability

Axioms immediately imply several simple, but important, properties of probabilities

Assume that P is a probability measure defined on a sample space Ω . Then

1. $P(F^c) = 1 - P(F)$
2. $P(F) \leq 1$
3. $P(\emptyset) = 0$

Proofs:

1. $P(F^c) = 1 - P(F)$

$F \cup F^c = \Omega$ implies $P(F \cup F^c) = 1$ (Axiom 2). $F \cap F^c = \emptyset$ implies $1 = P(F \cup F^c) = P(F) + P(F^c)$ (Axiom 3).

2. $P(F) \leq 1$

$P(F) = 1 - P(F^c) \leq 1$ (Axiom 1 and (a) above).

3. $P(\emptyset) = 0$

Axiom 2 and (1) above.

4. $P(G) = \sum_i P(G \cap F_i)$

4. "Total Probability Theorem" If $\{F_i; i = 1, 2, \dots\}$ is a partition of Ω , i.e., if $F_i \cap F_k = \emptyset$ when $i \neq k$ and $\bigcup_i F_i = \Omega$, then

$$P(G) = \sum_i P(G \cap F_i) \quad (1)$$

5. If $F \subset G$, then $P(F) \leq P(G)$.

6. $P(F \cup G) = P(F) + P(G) - P(F \cap G)$.

7. $P(F \cup G) \leq P(F) + P(G)$. (Union bound or Bonferroni inequality.)

8. $P\left(\bigcup_{i=1}^n F_i\right) \leq \sum_{i=1}^n P(F_i)$

$$\begin{aligned} P(G) &= P(G \cap \Omega) \text{ set theory} \\ &= P(G \cap \left(\bigcup_i F_i\right)) \text{ set theory} \\ &= P\left(\bigcup_i (G \cap F_i)\right) \text{ set theory} \\ &= \sum_i P(G \cap F_i). \text{ Axiom 3} \end{aligned}$$

5. If $F \subset G$, then $P(F) \leq P(G)$

From set theory, if $F \subset G$, then $G = F \cup (G \cap F^c)$. Since this is a union of disjoint events, $P(G) = P(F) + P(G \cap F^c) \geq P(F)$ from Axiom 1.

$$6. P(F \cup G) = P(F) + P(G) - P(F \cap G)$$

Can rewrite a union as a disjoint union: $F \cup G = F \cup (G \cap F^c)$ so that $P(F \cup G) = P(F) + P(G \cap F^c)$. Also know that

$$\begin{aligned} P(G) &= P(G \cap \Omega) \text{ set theory} \\ &= P(G \cap (F \cup F^c)) \text{ set theory} \\ &= P((G \cap F) \cup (G \cap F^c)) \text{ set theory} \\ &= P(G \cap F) + P(G \cap F^c) \text{ Axiom 3} \end{aligned}$$

which the previous formula proves the result.

$$7. P(F \cup G) \leq P(F) + P(G)$$

Follows from the previous property and Axiom 1.

$$8. P\left(\bigcup_{i=1}^n F_i\right) \leq \sum_{i=1}^n P(F_i)$$

Just iterate on previous result:

$$P\left(\bigcup_{i=1}^n F_i\right) \leq P(F_1) + \underbrace{P\left(\bigcup_{i=2}^n F_i\right)}_{\leq P(F_2) + P\left(\bigcup_{i=2}^n F_i\right)}$$

etc.