Probability Theory

Bayes' Rule: $\Pr\{Y=y|X=x\} = \frac{\Pr\{X=x,Y=y\}}{\Pr\{X=x\}}$
Expectation: $\mathbf{E}[g(X)] = \sum_{\text{all } x} g(x) \Pr\{X=x\}$
Mean: $\mu_X = \mathbf{E}[X]$
Variance: $\sigma_X^2 = \mathbf{E}\left[(X-\mu_X)^2\right]$
Covariance: $\text{Cov}(X,Y) = \mathbf{E}[(X-\mu_X)(Y-\mu_Y)]$
Correlation: $\rho(X,Y) = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}$
Autocorrelation: $\phi_X(\tau) = \mathbf{E}[X(t)X(t+\tau)]$
Properties: Average power $= \phi_X(0)$
$\phi_X(-\tau) = \phi_X(\tau)$
$|\phi_X(\tau)| \le \phi_X(0)$
Cross-correlation: $\phi_{XY}(t_1,t_2) = \mathbf{E}[X(t_1)Y(t_2)]$
PSD: $\Phi_X(f) = \int_{-\infty}^{\infty} \phi_X(\tau) e^{-j2\pi f\tau}\, d\tau$

Gaussian Distribution:
- probability density function:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} \exp\left\{-\frac{(x-\mu_X)^2}{2\sigma_X^2}\right\}$$

- error function

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\, du$$

$$Q(x) = \tfrac{1}{2}\text{erfc}\left(\frac{x}{\sqrt{2}}\right)$$

- if two Gaussian random variables are uncorrelated, they are also independent.
- any linear combination of two or more Gaussian random variables results in another Gaussian random variable.

Signal Space Concepts

Inner product: $\langle s_k(t), s_l(t)\rangle \triangleq \int_a^b s_k(t)s_l^*(t)\, dt$
Triangle inequality:
$\|s_k(t)+s_l(t)\| \le \|s_k(t)\| + \|s_l(t)\|$

Norm: $\|s_k(t)\| \triangleq \sqrt{\langle s_k(t), s_k(t)\rangle}$
Cauchy-Schwartz inequality;
$|\langle s_k(t), s_l(t)\rangle| \le \|s_k(t)\|\ \|s_l(t)\|$

Information Theory

Entropy: $H(X) = \sum_{k=0}^{K-1} P(x_k)\log_2 \frac{1}{P(x_k)}$
Binary Entropy Function: $h(p) = p\log_2 \frac{1}{p} + (1-p)\log_2 \frac{1}{(1-p)}$
Conditional Entropy: $H(X|Y) = \sum_{k=0}^{K-1}\sum_{l=0}^{L-1} P(x_k,y_l)\log_2 \frac{1}{P(x_k|y_l)}$
Average Mutual Information: $I(X;Y) = H(X) - H(X|Y) = H(Y) - H(Y|X)$
Channel Capacity: $C = \max_{\{P(X_k)\}} I(X;Y)$
Differential Entropy: $H(X) = \int_{-\infty}^{\infty} f_X(x)\log_2 \frac{1}{f_X(x)}\, dx$
Conditional Differential Entropy: $H(X|Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\log_2 \frac{1}{f_{X|Y}(x|y)}\, dx\, dy$
The Fundamental Inequality:
Let $\{p_i\}$ and $\{s_i\}$ be any two sets of N real numbers with $p_i \ge 0$, $s_i \ge 0$, $\sum_{i=0}^{N-1} p_i = 1$, and $\sum_{i=0}^{N-1} s_i = 1$, then $\sum_{i=0}^{N-1} p_i \ln \frac{1}{p_i} \le \sum_{i=0}^{N-1} p_i \ln \frac{1}{s_i}$ with equality iff $s_i = p_i$ for all i.

Linear Block Codes

Def^n: An (n,k) binary code, $\mathcal{C}$, consists of a set of 2^k binary code words, each of length n bits, and a mapping function function between message words and code words.

Def^n: An (n,k) *linear block code* is defined by a $k \times n$ generator matrix, $\underline{\underline{G}}$, such that the code word $\underline{c}$ for message $\underline{m}$ is obtained from $\underline{c} = \underline{m}\ \underline{\underline{G}}$

Hamming weight: number of 1's in a word.
Minimum weight:
$w_{\min} = \min\{w_H(\underline{c}) \mid \underline{c} \in \mathcal{C}, \underline{c} \ne \underline{0}\}$
Hamming distance: $d_H(\underline{a},\underline{b}) = w_H(\underline{a} \oplus \underline{b})$
Minimum distance:
$d_{\min} = \min\{d_H(\underline{a},\underline{b}) \mid \underline{a},\underline{b} \in \mathcal{C}, \underline{a} \ne \underline{b}\}$
For all linear block codes, $d_{\min} = w_{\min}$.

Code Rate: $R = \frac{k}{n}$.
Parity check matrix: $\underline{\underline{G}}\ \underline{\underline{H}}^T = \underline{\underline{0}}$
if $\underline{\underline{G}} = \left[\underline{\underline{I}}_k \mid \underline{\underline{P}}\right]$ then $\underline{\underline{H}} = \left[\underline{\underline{P}}^T \mid \underline{\underline{I}}_{n-k}\right]$
Syndrome: $\underline{s} = \underline{r}\ \underline{\underline{H}}^T$
Random-error detecting capability: $d_{\min} - 1$.
Random-error correcting capability: $t = \left\lfloor \frac{d_{\min}-1}{2} \right\rfloor$

Trigonometric Identities

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\sin A \sin B = \tfrac{1}{2}[\cos(A-B) - \cos(A+B)]$$
$$\cos A \cos B = \tfrac{1}{2}[\cos(A-B) + \cos(A+B)]$$
$$\sin A \cos B = \tfrac{1}{2}[\sin(A+B) + \sin(A-B)]$$
$$\cos A \sin B = \tfrac{1}{2}[\sin(A+B) - \sin(A-B)]$$
$$\sin 2A = 2 \sin A \cos A$$
$$\cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A = \cos^2 A - \sin^2 A$$
$$\sin^2 A = \tfrac{1}{2}(1 - \cos 2A)$$
$$\cos^2 A = \tfrac{1}{2}(1 + \cos 2A)$$
$$\sin A = \frac{1}{j2}\left(e^{jA} - e^{-jA}\right)$$
$$\cos A = \frac{1}{2}\left(e^{jA} + e^{-jA}\right)$$
$$e^{\pm jA} = \cos A \pm j \sin A$$

Miscellaneous Identities

$$\sum_{m=-\infty}^{\infty} e^{-j2\pi f m T} = \frac{1}{T}\sum_{n=-\infty}^{\infty} \delta(f - \frac{n}{T})$$

Definite Integrals

$$\int_0^\infty \frac{\sin ax}{x}\,dx = \begin{cases} \pi/2 & a > 0 \\ 0 & a = 0 \\ -\pi/2 & a < 0 \end{cases}$$
$$\int_0^x \frac{\sin au}{u}\,du = \mathrm{Si}(x)$$
$$\int_0^\infty \frac{\sin^2 ax}{x^2}\,dx = |a|\pi/2$$
$$\int_0^\infty e^{-ax^2}\,dx = \tfrac{1}{2}\sqrt{\pi/a}$$
$$\int_0^\infty x e^{-ax^2}\,dx = \frac{1}{2a}$$
$$\int_0^\infty x^2 e^{-ax^2}\,dx = \frac{1}{4a}\sqrt{\pi/a}$$
$$\int_0^x \frac{2}{\sqrt{\pi}} e^{-u^2}\,du = \mathrm{erf}\,(x)$$
$$\int_{-\infty}^\infty e^{j2\pi f t}\,dt = \delta(f)$$

Indefinite Integrals

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b)$$
$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b)$$
$$\int \sin^2 ax\,dx = \frac{x}{2} - \frac{\sin 2ax}{4a}$$
$$\int \cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$$
$$\int \sin ax \cos ax\,dx = \frac{1}{2a}\sin^2 ax$$
$$\int \sin ax \sin bx\,dx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)}$$
$$\int \cos ax \cos bx\,dx = \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)}$$
$$\int \sin ax \cos bx\,dx = -\frac{\cos(a-b)x}{2(a-b)} - \frac{\cos(a+b)x}{2(a+b)}$$
$$\int \cos ax \sin bx\,dx = \frac{\cos(a-b)x}{2(a-b)} - \frac{\cos(a+b)x}{2(a+b)}$$
$$\int x \sin ax\,dx = \frac{1}{a^2}(\sin ax - ax\cos ax)$$
$$\int x \cos ax\,dx = \frac{1}{a^2}(\cos ax + ax\sin ax)$$
$$\int x^2 \sin ax\,dx = \frac{1}{a^3}(2ax\sin ax + 2\cos ax - a^2x^2\cos ax)$$
$$\int x^2 \cos ax\,dx = \frac{1}{a^3}(2ax\cos ax - 2\sin ax + a^2x^2\sin ax)$$
$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}$$
$$\int x e^{ax}\,dx = \frac{1}{a^2}e^{ax}(ax - 1)$$
$$\int x^2 e^{ax}\,dx = \frac{1}{a^3}e^{ax}(a^2x^2 - 2ax + 2)$$
$$\int e^{ax}\sin bx\,dx = \frac{1}{a^2+b^2}e^{ax}(a\sin bx - b\cos bx)$$
$$\int e^{ax}\cos bx\,dx = \frac{1}{a^2+b^2}e^{ax}(a\cos bx + b\sin bx)$$
$$\int \left[\frac{\sin ax}{x}\right]^2 dx = a\int \frac{\sin 2ax}{x}\,dx - \frac{\sin^2 ax}{x}$$

Properties of the Fourier Transform

Operation	$h(t)$	$H(f)$
Linearity	$a_1h_1(t)+a_2h_2(t)$	$a_1H_1(f)+a_2H_2(f)$
Complex conjugate	$h^*(t)$	$H^*(-f)$
Scaling	$h(\alpha t)$	$\frac{1}{\lvert\alpha\rvert}H(\frac{f}{\lvert\alpha\rvert})$
Delay	$h(t-t_0)$	$H(f)e^{-j2\pi f t_0}$
Frequency translation	$h(t)e^{j2\pi f_0 t}$	$H(f-f_0)$
Amplitude modulation	$h(t)\cos(2\pi f_0 t)$	$\frac{1}{2}H(f-f_0)+\frac{1}{2}H(f+f_0)$
Time convolution	$\int_{-\infty}^{\infty} h_1(\tau)h_2(t-\tau)\,d\tau$	$H_1(f)H_2(f)$
Frequency convolution	$h_1(t)h_2(t)$	$\int_{-\infty}^{\infty} H_1(u)H_2(f-u)\,du$
Duality	$H(t)$	$h(-f)$
Time differentiation	$\frac{d}{dt}h(t)$	$j2\pi f H(f)$
Time integration	$\int_{-\infty}^{t} h(\tau)\,d\tau$	$\frac{1}{j2\pi f}H(f)+\frac{H(0)}{2}\delta(f)$

Some Fourier Transform Pairs

$h(t)$	$\rightarrow$	$H(f)$
$e^{-at}u(t)$	$\rightarrow$	$\dfrac{1}{a+j2\pi f}$
$te^{-at}u(t)$	$\rightarrow$	$\dfrac{1}{(a+j2\pi f)^2}$
$e^{-a\lvert t\rvert}$	$\rightarrow$	$\dfrac{2a}{a^2+(2\pi f)^2}$
$e^{-t^2/(2\sigma^2)}$	$\rightarrow$	$\sqrt{2\pi\sigma^2}e^{-2\pi^2 f^2\sigma^2}$
$u(t)$	$\rightarrow$	$\frac{1}{2}\delta(f)+\dfrac{1}{j2\pi f}$
$\delta(t-t_0)$	$\rightarrow$	$e^{-j2\pi f t_0}$
$\dfrac{\sin 2\pi W t}{2\pi W t}$	$\rightarrow$	$\dfrac{1}{2W}\text{rect}(\dfrac{f}{2W})$
$\text{rect}(\dfrac{t}{T})$	$\rightarrow$	$T\dfrac{\sin \pi f T}{\pi f T}$
$\sum_{m=-\infty}^{\infty}\delta(t-mT)$	$\rightarrow$	$\dfrac{1}{T}\sum_{n=-\infty}^{\infty}\delta(f-\dfrac{n}{T})$