Binary Signaling

(inspired from the notes of Prof. Pasupathy, U of Toronto)

$0 \to s_0(t)$

$1 \to s_1(t)$

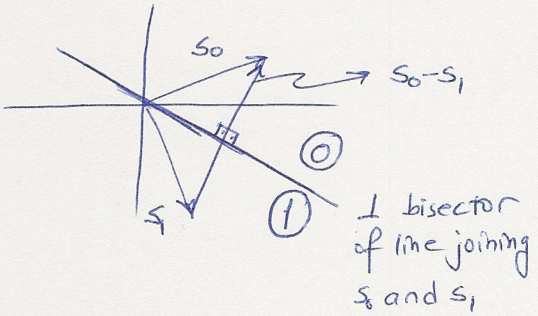
$\|s_0\|^2 = \|s_1\|^2 = E_b$

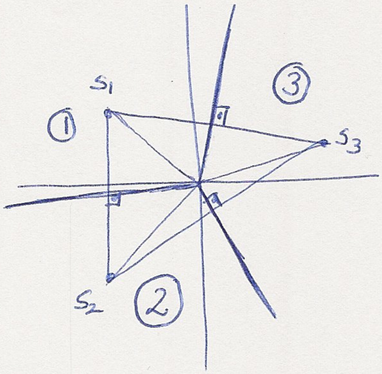
Optimal Decision Rule: $(r, s_0) \underset{1}{\overset{0}{\gtrless}} (r, s_1)$

$(r, s_0 - s_1) \underset{1}{\overset{0}{\gtrless}} 0$

decision boundary: $(r, s_0 - s_1) = 0$

$\Rightarrow r \perp s_0 - s_1$



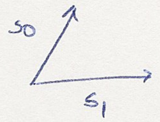


$P_e = P(1|0)P_0 + P(0|1)P_1$

if $P_0 = P_1 = \frac{1}{2} \to P_e = \frac{1}{2}\Big(\underbrace{P(1|0)}_{P_{1|0}} + \underbrace{P(0|1)}_{P_{0|1}}\Big)$

$P_{1|0} = P\Big((r, s_0 - s_1) | s_0 < 0\Big)$

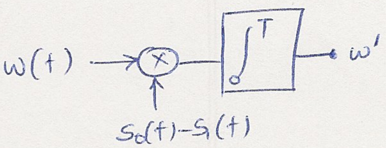
$= P\Big((s_0 + w, s_0 - s_1) < 0\Big) = P\Big(\underbrace{(s_0, s_0)}_{E_b} - \underbrace{(s_0, s_1)}_{\rho E_b} + \underbrace{(w, s_0 - s_1)}_{w'} < 0\Big)$

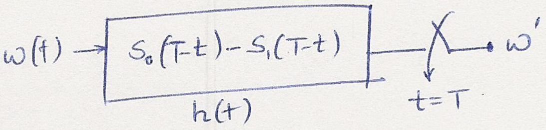


$(s_0, s_1) = \|s_0\| \|s_1\| \underbrace{\cos\theta}_{\rho}$

$\rho = \frac{(s_0, s_1)}{\|s_0\| \|s_1\|}$

$w' = (w, s_0 - s_1)$





$w(t)$: gaussian $\rightarrow$ filter output: gaussian $\rightarrow$ w': gaussian r.v.

$w(t)$: zero-mean $\longrightarrow$ $E(w')=0$

$$\sigma_{w'}^2 = E\left[w'^2\right] = \int_{-\infty}^{\infty} S_w(f)\,|H(f)|^2\,df = \frac{N_0}{2}\int_{-\infty}^{\infty}|H(f)|^2\,df$$

$$= \frac{N_0}{2}\int_{-\infty}^{\infty}|h(t)|^2\,dt = \frac{N_0}{2}\int_{-\infty}^{\infty}\left(s_0(T-t)-s_1(T-t)\right)^2 dt$$

$$= \frac{N_0}{2}\int_{-\infty}^{\infty}\left(s_0(t)-s_1(t)\right)^2 dt = \frac{N_0}{2}\left(\|s_0\|^2+\|s_1\|^2-2(s_0,s_1)\right)$$

$$= \frac{N_0}{2}\left(2E_b-2\rho E_b\right)$$

$$= N_0 E_b(1-\rho)$$

$$P_{1|0} = P\left(E_b(1-\rho)+w'<0\right)$$

$$= P\left(w' < -E_b(1-\rho)\right)$$

$\hookrightarrow G\left(0;\sigma^2 = N_0 E_b(1-\rho)\right)$

$$\frac{2}{\sqrt{\pi}}\int_{-\infty}^{-k} e^{-u^2}\,du = \mathrm{erfc}(k)$$

$$= \int_{-\infty}^{-E_b(1-\rho)} \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{w'^2}{2\sigma^2}}\,dw'$$

$$\frac{w'^2}{2\sigma^2} = u^2$$

$$= \int_{-\infty}^{-\frac{E_b(1-\rho)}{\sqrt{2}\sigma}} \frac{1}{\sqrt{2\pi}\,\sigma}\,\sqrt{2}\sigma\, e^{-u^2}\,du = \frac{1}{2}\,\mathrm{erfc}\left[\frac{E_b(1-\rho)}{\sqrt{2}\sqrt{N_0 E_b(1-\rho)}}\right]$$

$$\boxed{\begin{aligned} &= \frac{1}{2}\,\mathrm{erfc}\sqrt{\frac{E_b(1-\rho)}{2N_0}} \\ &= \frac{1}{2}\,\mathrm{erfc}\left(\frac{d_{01}}{2\sqrt{N_0}}\right) \end{aligned}}$$

$$P_e = \frac{1}{2}\left(P_{1|0}+P_{0|1}\right)$$

$$= \frac{1}{2}\,\mathrm{erfc}\sqrt{\frac{E_b(1-\rho)}{2N_0}} = \frac{1}{2}\,\mathrm{erfc}\left(\frac{d_{01}}{2\sqrt{N_0}}\right)$$

Prob. of Error Analysis in Binary Signaling

Baseband or bandpass

Equally-likely bits ($p_0 = p_1 = 1/2$)

valid for non equal energy case as well!

$$\boxed{P_e = \frac{1}{2}\,\text{erfc}\sqrt{\frac{E_b(1-\rho_{01})}{2N_0}} = \frac{1}{2}\,\text{erfc}\left(\frac{d_{01}}{2\sqrt{N_0}}\right)}$$

in the presence of optimum detection rule

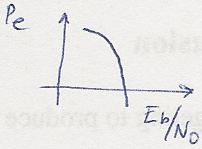
minimize P_e

↓

maximize argument of erfc → minimize ρ_{01} = maximize d_{01}

↓

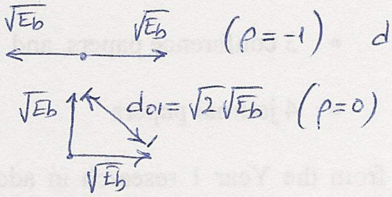
$\rho_{01} = -1$ → antipodal signaling



log-log scale

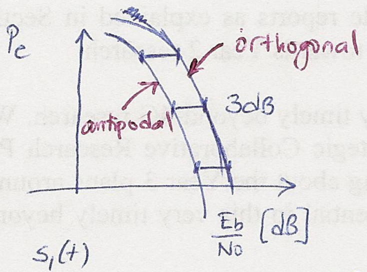
$$P_e \xrightarrow{\text{antipodal}} \frac{1}{2}\,\text{erfc}\sqrt{\frac{E_b}{N_0}}$$

$$\xrightarrow{\perp} \frac{1}{2}\,\text{erfc}\sqrt{\frac{E_b}{2N_0}}$$



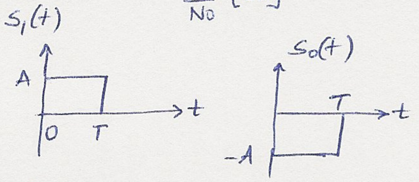
$(\rho = -1)$ $d_{01} = 2\sqrt{E_b}$

$d_{01} = \sqrt{2}\sqrt{E_b}$ $(\rho = 0)$



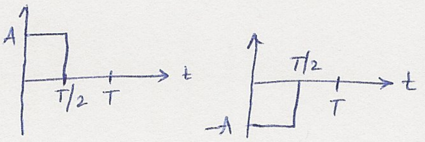
Antipodal

NRZ



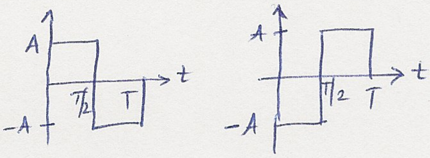
$E_b = A^2T$ $P_e = \frac{1}{2}\,\text{erfc}\sqrt{\frac{E_b}{N_0}}$

RZ



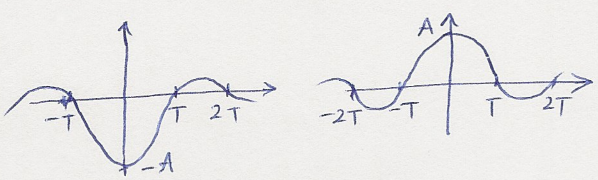
$E_b = \frac{A^2T}{2}$ (3dB worse than NRZ due to lack of energy!)

Manchester code



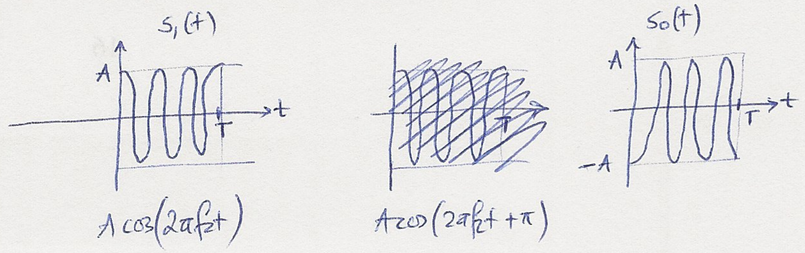
$E_b = A^2T$

sinc



$E_b = A^2T$

BPSK

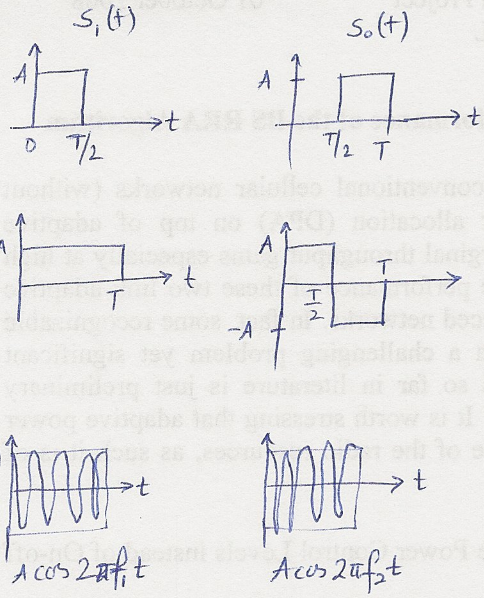


$A\cos(2\pi f_c t)$ $A\cos(2\pi f_c t + \pi)$

$E_b = \frac{A^2T}{2}$

Orthogonal

PPM



$E_b = \frac{A^2T}{2}$

$P_e = \frac{1}{2}\,\mathrm{erfc}\sqrt{\frac{E_b}{2N_0}}$

$E_b = A^2T$

$E_b = \frac{A^2T}{2}$

B⊥FSK: $s_1(t) = \cos 2\pi f_2 t$ $s_0(t) = \cos\left(2\pi(f_2+\Delta f)t\right)$

$\rho_{10} = 0 \rightarrow \int_0^T s_1(t)s_0(t)\,dt = 0$

$\cos x \cos y = \frac{1}{2}\left(\cos(x+y) + \cos(x-y)\right)$

$$(s_1,s_0) = \int_0^T \cos(2\pi f_2 t) \times \cos\left(2\pi(f_2+\Delta f)t\right)dt$$

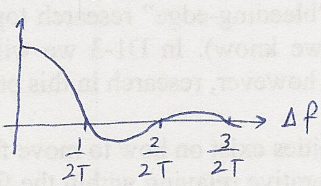
$$= \frac{1}{2}\int\left[\cos\left(2\pi(2f_2+\Delta f)t\right) + \cos(2\pi\Delta f t)\right]dt$$

$$= \frac{1}{2}\,\frac{\overset{-1<\ <1}{\sin\left(2\pi(2f_2+\Delta f)t\right)}}{\underbrace{2\pi(2f_2+\Delta f)}_{\text{very large}}}\Bigg|_0^T + \frac{1}{2}\,\frac{\sin(2\pi\Delta f t)}{2\pi\Delta f}\Bigg|_0^T$$

$T_c \gg T_c = \frac{1}{f_c}$

$$\approx \frac{T}{2}\,\frac{\sin 2\pi\Delta f T}{2\pi\Delta f T} = \frac{T}{2}\,\mathrm{sinc}(2\Delta f T)$$

$\frac{\sin \pi x}{\pi x} = \mathrm{sinc}\, x$



if $\Delta f = \frac{k}{2T} \rightarrow \rho_{10} \approx 0$

$\rightarrow$ ⊥FSK

$\Delta f = \frac{1}{2T} \rightarrow$ MSK (minimum shift keying)

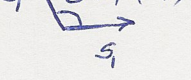
$s_1(t) = A\cos 2\pi f_2 t$

$s_0(t) = A\cos\left(2\pi(f_2+\Delta f)t\right)$

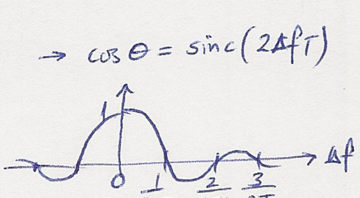
$E_1 = E_0 = \frac{A^2T}{2}$

$(s_1, s_0) = \frac{A^2T}{2}\,\mathrm{sinc}(2\Delta f T) = \sqrt{E_1}\sqrt{E_0}\cos\theta$

$\rightarrow \cos\theta = \mathrm{sinc}(2\Delta f T)$



$\theta = \pm 90°$ when $\Delta f = \frac{k}{2T}$



Non-Equal-Energy Binary Signals

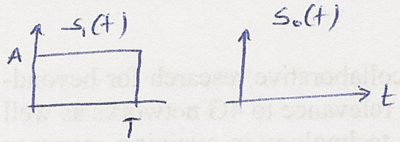
$$(r, s_0 - s_1) \overset{0}{\underset{1}{\gtrless}} \underbrace{\frac{E_0 - E1}{2}}_{\text{threshold revised!}} \leftarrow (r, s_0) - \frac{E_0}{2} \overset{0}{\underset{1}{\gtrless}} (r, s_1) - \frac{E_1}{2}$$

- equally likely

If non equally likely, 1) then optimal threshold has to be found

2) $P_e = p_0 P_{e|0} + p_1 P_{e|1}$

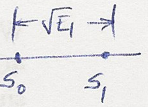
Ex: OOK



$$\underbrace{E_1 = A^2T \qquad E_0 = 0}$$

$$E_{av} = \frac{1}{2} E_1 + \frac{1}{2} E_0 = \frac{E_1}{2}$$

$$\perp \rightarrow P_e = \frac{1}{2} \operatorname{erfc}\left(\frac{d_{01}}{2\sqrt{N_0}}\right) = \frac{1}{2} \operatorname{erfc}\sqrt{\frac{E_1}{4N_0}} = \frac{1}{2} \operatorname{erfc}\sqrt{\frac{E_{av}}{2}}$$



6 dB loss wrt antipodal signaling with $E = A^2T$

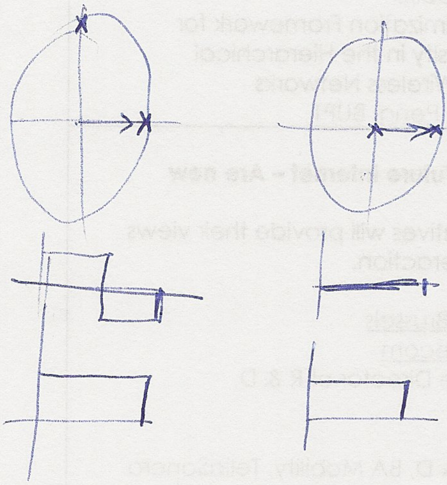
3dB due to orthogonality + 3dB due to half energy

OOK

$s_1 = 0$ $s_0 \neq 0$

$E_{av} = \frac{E_0}{2}$

$P_e = \frac{1}{2} \cancel{e}k \sqrt{\frac{E_{av}}{2N_0}} = \frac{1}{2} k \sqrt{\frac{E_0}{4N_0}}$



E_r

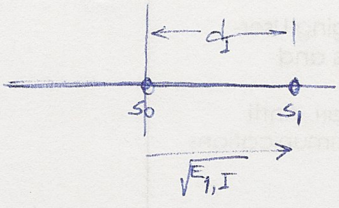
$(r, s_0) \gtrless^{0}_{1} (r, s_1)$ qu

$(r, s_0) - \frac{E_0}{2} \gtrless^{0}_{1} (r, s_1) - \frac{E_1}{2}$

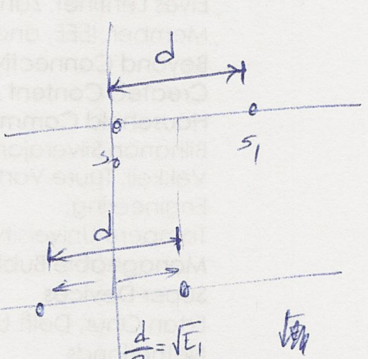
$d = \sqrt{E_1}$

$E_{av} =$ [illegible] d^2

①



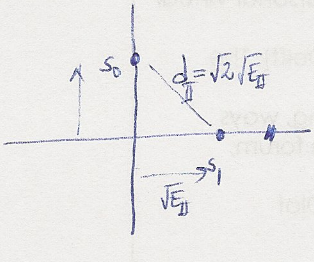
$E_{av,I} = \frac{1}{2} E_{1,I}$
$= \frac{1}{2}$



[illegible]

$E_{av} = \frac{1}{2} \frac{d^2}{4} + \frac{1}{2} \frac{d^2}{4} = \frac{d^2}{4}$

(II)



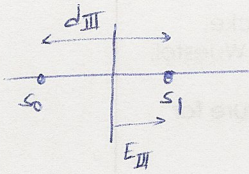
$d_{II} = \sqrt{2}\sqrt{E_{II}}$

$E_{av,II} = \frac{1}{2} E_{1,II} + \frac{1}{2} E_{0,II}$
$= \cancel{\frac{d}{4}} E_{II}$

$\overset{d_{II}}{\sqrt{2}\sqrt{E_{II}}} = \overset{d_I}{\sqrt{E_{1,I}}}$

$2E_{II} = E_{1,I} \longrightarrow \left\{ E_{av,I} = E_{II} = E_{av,II} \right.$

(III)

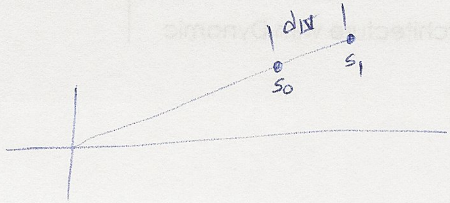


$E_{av,III} = E_{III}$

$d_{III} = 2\sqrt{E_{III}} = \sqrt{2}\sqrt{E_{II}} = \sqrt{E_{1,I}}$

$4E_{III} = 2E_{II} = E_{1,I} = 2E_{av,I}$

(IV)



if $d_{IV} = d_{III} (= d_{II} = d_I) \rightarrow$ same P_e, but much higher E_{av}

or, for the same E_{av}, much ~~[illegible]~~ higher P_e!

28 Oct 08