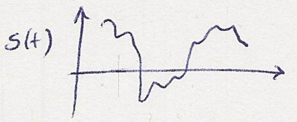
Signal Space Analysis (Ch 5) (big topic)

geometric representation of signals

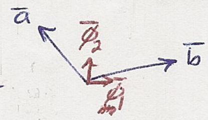
(Kotel'nikov 1947, Molotov Energy Institute, Moscow)
(Wozencraft and Jacobs 1965)



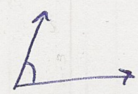
$\Rightarrow$

vector in n-dimensional space

* How many dimensions are needed to represent a set of signals ?
  a set of vectors ?



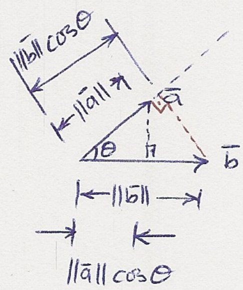
* Basis vectors, decomposition
* Dot product, angle between two vectors (signals) $\bar{a}.\bar{b}$
  length



Linear Algebra

length : $\|\bar{a}\|$ norm

dot product, inner product



$\bar{a}.\bar{b} = \|\bar{a}\|\,\|\bar{b}\|\cos\theta$

$\cos\theta = \dfrac{\bar{a}.\bar{b}}{\|a\|\,\|b\|}$

$\theta = \arccos \dfrac{\bar{a}.\bar{b}}{\|\bar{a}\|\,\|\bar{b}\|}$

$\bar{a}$, $\bar{b}$ colinear vectors $\rightarrow \theta = 0$

$\bar{a}.\bar{b} = \|\bar{a}\|\,\|\bar{b}\|$ $\rightarrow$ length : norm : $\|\bar{a}\| = \sqrt{\bar{a}.\bar{a}}$

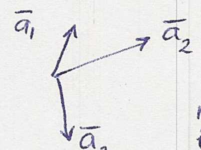
$\bar{a}$, $\bar{b}$ orthogonal vectors $\rightarrow \theta = 90$

$\bar{a}.\bar{b} = 0$

$\bar{a}$, $\bar{b}$ orthonormal vectors $\rightarrow \theta = 90°$

$\|\bar{a}\| = \|\bar{b}\| = 1$

$\{\bar{a}_i\}_{i=1}^{N}$: linearly independent $\rightarrow \sum_{i=1}^{N} \alpha_i \bar{a}_i = 0$ iff $\alpha_i = 0, \forall i$



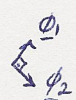
not linearly independent

orthonormal basis functions: $\{\underline{\phi}_i\} : \underline{\phi}_i . \underline{\phi}_j = 0, \forall i \neq j$

$\underline{\phi}_i . \underline{\phi}_i = 1, \forall i$

A set of N orthonormal basis functions span an N dimensional space; i.e., every vector in that space can be written as $\underline{a} = \sum \alpha_i \underline{\phi}_i$ with at least one $\alpha_i \neq 0$.

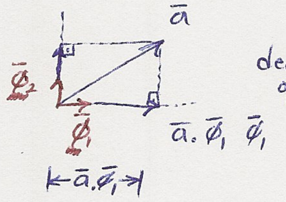
Ex:



Note: there are infinitely many sets of N-element orthonormal basis functions that span an N-dim space.

Ex:

projection of a vector on another one



decomposition of a vector

Gram-Schmidt Orthogonalization Procedure

component along a particular axis

$\alpha_i = \bar{a}_i \cdot \bar{\phi}_i$

$\{\bar{a}_i\}_{i=1}^{M}$

$\downarrow$

$\{\bar{\phi}_i\}_{i=1}^{N \le M}$

1) $\bar{\phi}_1 = \dfrac{\bar{a}_1}{\|\bar{a}_1\|}$

2) $\bar{w}_2 = \bar{a}_2 - \overbrace{\bar{a}_2 \cdot \bar{\phi}_1}^{a_{21}}\ \bar{\phi}_1$

$\bar{\phi}_2 = \dfrac{\bar{w}_2}{\|\bar{w}_2\|}$

3) $\bar{w}_3 = \bar{a}_3 - \underbrace{a_{31}}_{\bar{a}_3\cdot\bar{\phi}_1}\bar{\phi}_1 - \underbrace{a_{32}}_{\bar{a}_3\cdot\bar{\phi}_2}\bar{\phi}_2$

$\bar{\phi}_3 = \dfrac{\bar{w}_3}{\|\bar{w}_3\|}$

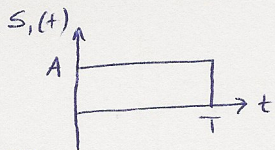
$\vdots$

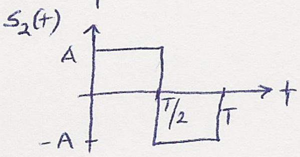
4) $\bar{\phi}_N = \dfrac{\bar{w}_N}{\|\bar{w}_N\|}$, where $\bar{w}_N = \bar{a}_N - \sum_{i=1}^{N-1} \underbrace{a_{Ni}}_{\bar{a}_N\bar{\phi}_i}\bar{\phi}_i$

Note $N \le M$

$\bar{a}_i = \sum_{i=1}^{N} \alpha_i \bar{\phi}_i \qquad i=1,\dots,M$

$\|\bar{a}_i\|^2 = \sum_{i=1}^{N} \alpha_i^2$





$\int s_1(t)\, s_2(t)\, dt = 0$ ... inner product

orthogonal

$\int s_1^2(t)\,dt = E \rightarrow$ squared-norm $\rightarrow \sqrt{E}$: length

normal : unit-energy ($E=1$)

Inner product : $\int s_1(t)\, s_2(t)\, dt$

Norm : $\sqrt{E}$

Orthonormal basis functions : $\{\phi_i(t)\}_{i=1}^{N} \rightarrow \int \phi_i(t)\phi_j(t)\,dt = \delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i \ne j \end{cases}$

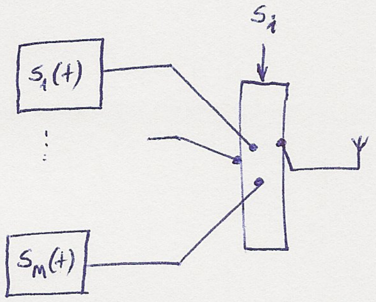
Decomposition of a set of signals $\{s_i(t)\}_{i=1}^{M}$ $\qquad s_i(t) = \sum_{j=1}^{N} s_{ij}\,\phi_j(t)$

$\hookrightarrow s_{ij} = \int s_i(t)\,\phi_j(t)\,dt$

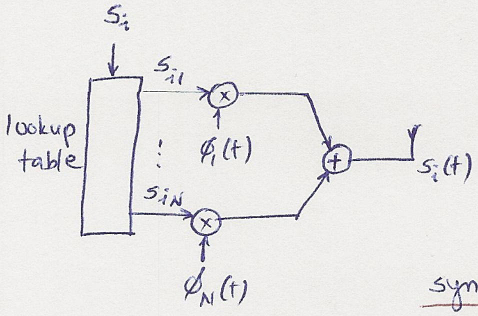
G-S Orthogonalization procedure

— Oct 7/08

signal generation



dial-up modems $M=1024$ } MQAM
wireless $M=64$



synthesizer

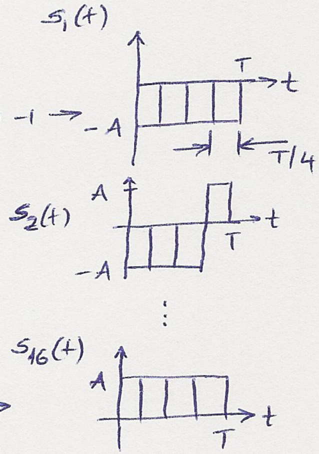
MQAM $\rightarrow$ N = 2 !

(goto end of p.4)

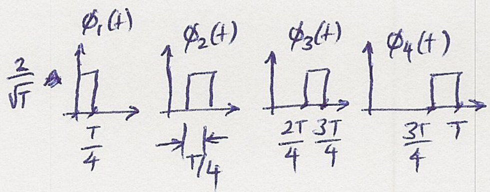
Ex2: 16 signals

−1 −1 −1 −1 →

1 1 1 1 →

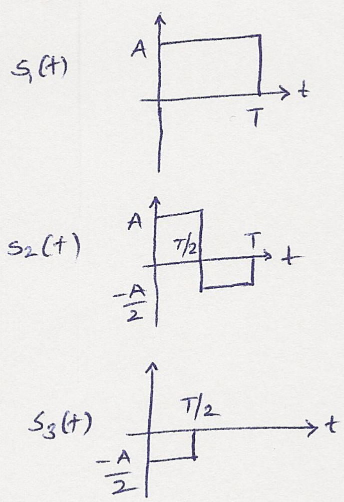


by inspection $\Longrightarrow$



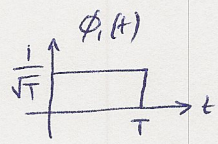
Note: G-S will give a different set of 4 orthonormal basis functions

Ex1: G-S

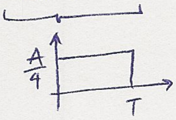


$$\phi_1(t) = \frac{s_1(t)}{\|s_1(t)\|}$$

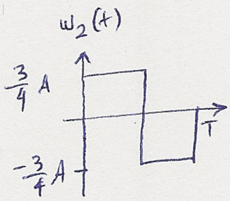
$$\|s_1(t)\| = \sqrt{A^2 T} \rightarrow$$



$$w_2(t) = s_2(t) - \underbrace{s_{21}\,\phi_1(t)}$$

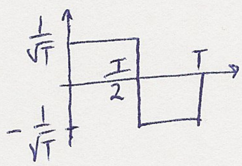


$$s_{21} = \int_0^T s_2(t)\,\phi_1(t)\,dt = \frac{1}{\sqrt{T}}\left(\frac{AT}{2} - \frac{AT}{4}\right) = \frac{A\sqrt{T}}{4}$$



$$\|w_2(t)\| = \frac{3}{4}A\sqrt{T}$$

$$\phi_2(t) = \frac{w_2(t)}{\|w_2(t)\|}$$



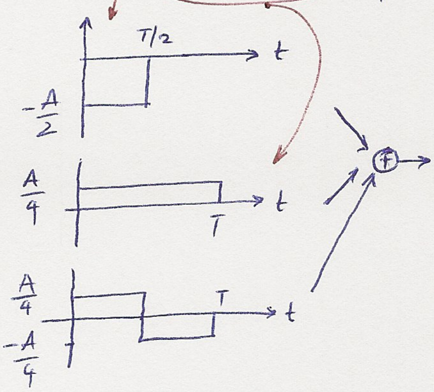
note: $w_2(t) \perp s_1(t)$
$\phi_2(t) \perp \phi_1(t)$

$$w_3(t) = s_3(t) - s_{31}\phi_1(t) - s_{32}\phi_2(t)$$

$$s_{31} = \int_0^{T/2} s_3(t)\phi_1(t)\,dt = -\frac{A}{2}\frac{1}{\sqrt{T}}\frac{T}{2} = -\frac{A\sqrt{T}}{4}$$

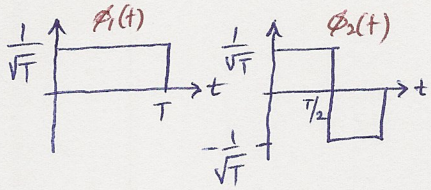
$$= s_3(t) + \frac{A\sqrt{T}}{4}\phi_1(t) + \frac{A\sqrt{T}}{4}\phi_2(t)$$

$$s_{32} = \int_0^{T/2} s_3(t)\phi_2(t)\,dt = -\frac{A}{2}\frac{1}{\sqrt{T}}\frac{T}{2} = -\frac{A\sqrt{T}}{4}$$



$\rightarrow 0$ ✓ $\Rightarrow \phi_3(t) = 0$

$\therefore \{s_i(t)\}_{i=1}^{3}$ can be represented by $\{\phi_j(t)\}_{j=1}^{2}$



$$s_1(t) = s_{11}\phi_1(t) + s_{12}\phi_2(t)$$

$$s_2(t) = s_{21}\phi_1(t) + s_{22}\phi_2(t)$$

$$s_3(t) = s_{31}\phi_1(t) + s_{32}\phi_2(t)$$

$$\begin{bmatrix} s_1(t) \\ s_2(t) \\ s_3(t) \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \\ s_{31} & s_{32} \end{bmatrix} \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \\ \cancel{\phi_3(t)} \end{bmatrix}$$

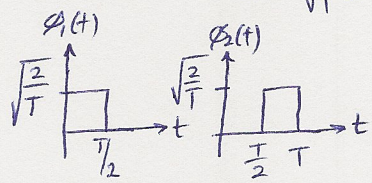
$$s_{11} = \int_0^T s_1(t)\phi_1(t)\,dt = A\sqrt{T}$$

$$s_{21} = \int_0^T s_2(t)\phi_1(t)\,dt = \frac{1}{\sqrt{T}}A\frac{T}{2} - \frac{1}{\sqrt{T}}\frac{A}{2}\frac{T}{2} = \frac{A\sqrt{T}}{4}$$

$$s_{12} = 0$$

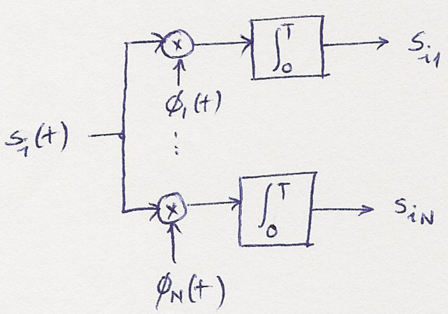
$$s_{22} = \frac{A}{\sqrt{T}}\frac{T}{2} + \frac{A}{2\sqrt{T}}\frac{T}{2} = \frac{3}{4}A\sqrt{T}$$

By inspection: Choose



A new set of $\{\phi_j\}_{j=1}^{2}$

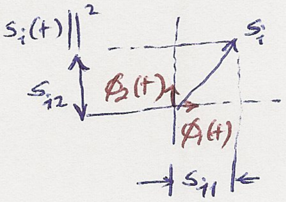
$$\begin{bmatrix} s_1(t) \\ s_2(t) \\ s_3(t) \end{bmatrix} = \begin{bmatrix} A\sqrt{\frac{T}{2}} & A\sqrt{\frac{T}{2}} \\ A\sqrt{\frac{T}{2}} & -\frac{A}{2}\sqrt{\frac{T}{2}} \\ -\frac{A}{2}\sqrt{\frac{T}{2}} & 0 \end{bmatrix} \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix}$$



analyzer

Vectorial representation of signals: $s_i(t) \equiv \bar{s}_i = \begin{bmatrix} s_{i1} \\ \vdots \\ s_{iN} \end{bmatrix}$

Note 1: $E_i = \sum_{j=1}^{N} s_{ij}^2 = \| s_i(t) \|^2$

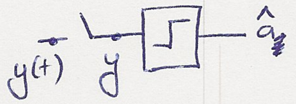


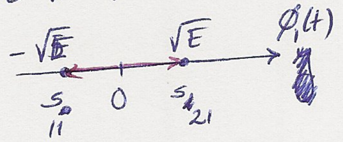
$\| s_i \|^2 = s_{i1}^2 + s_{i2}^2$

Note 2: $\underbrace{(s_i(t), s_j(t))}_{\text{signals of duration } T} = \int_0^T s_i(t) s_j(t)\, dt = \bar{s}_i^T \bar{s}_j = \sum_{k=1}^{N} s_{ik} s_{jk}$

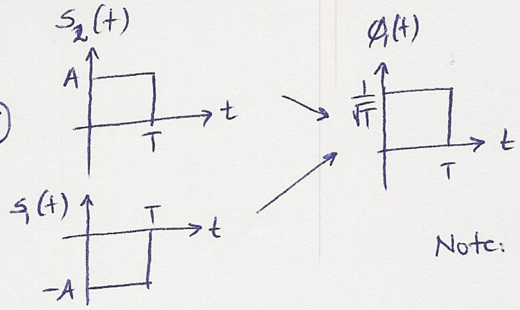
Note 3: $\| \bar{s}_i - \bar{s}_j \|^2 = \int_0^T (s_i(t) - s_j(t))^2 dt = \sum_{k=1}^{N} (s_{ik} - s_{jk})^2$

Sec 5.3 Conversion of the continuous AWGN channel into a Vector Channel





I



$s_1(t) = \overbrace{A\sqrt{T}}^{s_{11}} \phi_1(t) = \sqrt{E}\, \phi_1(t)$

$s_2(t) = \underbrace{-A\sqrt{T}}_{s_{21}} \phi_1(t) = -\sqrt{E}\, \phi_1(t)$

Note: $E_i = A^2 T$

$y = \pm\sqrt{E} + n$

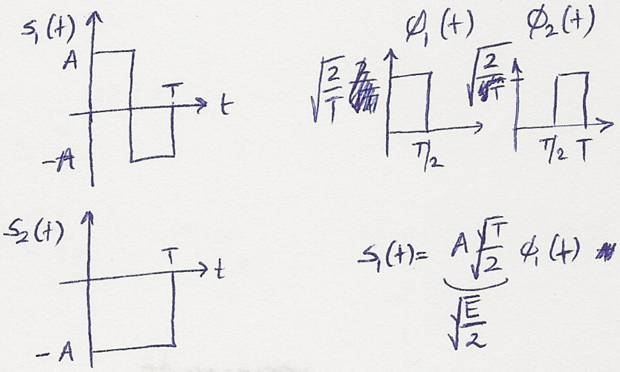
developed for antipodal signaling

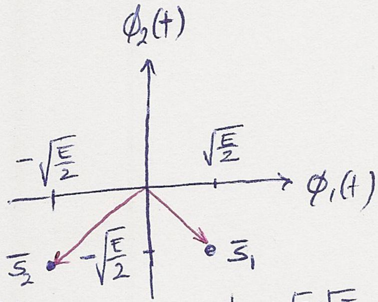
$P_e = \frac{1}{2} \operatorname{erfc} \sqrt{\frac{E}{N_0}}$

$\boxed{P_e = \frac{1}{2} \operatorname{erfc} \left(\frac{d}{2\sqrt{N_0}} \right)}$ general

$\Rightarrow d \uparrow \rightarrow P_e \downarrow$

Generalize this analysis





$d_{II} = \sqrt{2}\sqrt{E}$

Note $E_I = E_{II}$

$P_e = \frac{1}{2} \operatorname{erfc} \frac{\sqrt{2}\sqrt{E}}{2\sqrt{N_0}} = \frac{1}{2} \operatorname{erfc} \left(\sqrt{\frac{E}{2N_0}} \right)$

II

$s_1(t) = \underbrace{A\sqrt{\frac{T}{2}}}_{\sqrt{\frac{E}{2}}} \phi_1(t) - \underbrace{A\sqrt{\frac{T}{2}}}_{\sqrt{\frac{E}{2}}} \phi_2(t)$

$s_2(t) = -A\sqrt{\frac{T}{2}} \phi_1(t) - A\sqrt{\frac{T}{2}} \phi_2(t)$