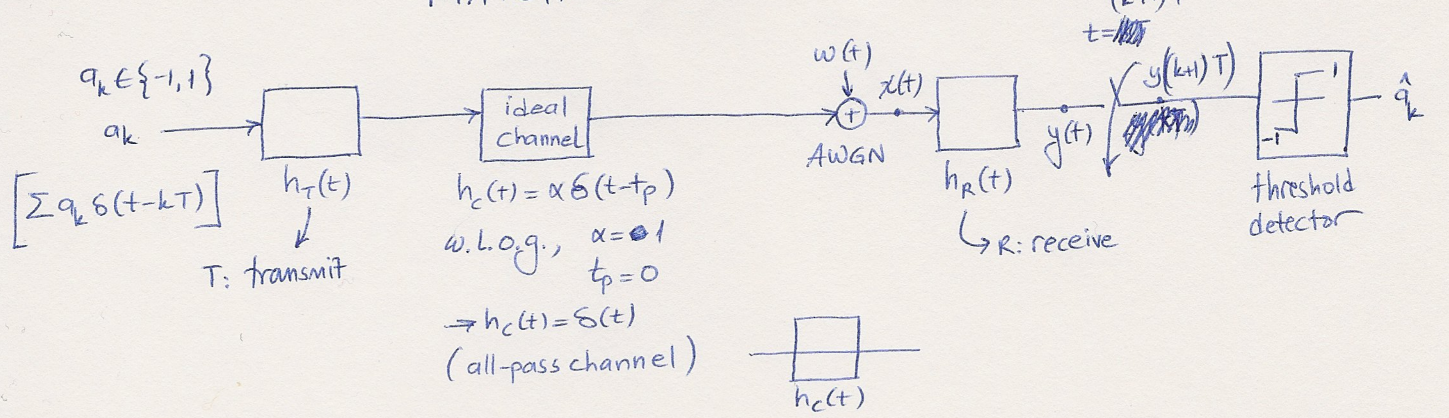
MATCHED FILTER



$h_c(t) = \alpha\,\delta(t-t_p)$

w.l.o.g., $\alpha = 1$, $t_p = 0$

$\rightarrow h_c(t) = \delta(t)$

(all-pass channel)

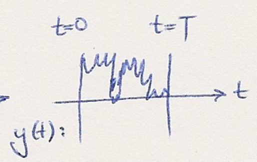
* Q: For a given transmit filter, $h_T(t)$, design the optimum receive filter, $h_R(t)$.

Optimization criterion: maximize SNR at the threshold detector input.

* $x(t) = \sum_k a_k h_T(t-kT) + w(t)$

$y(t) = x(t) * h_R(t)$

Observe $y(t)$, over $[kT, (k+1)T)$. W.L.o.g, $k=0$ →



$a_k = \pm 1 \quad \rightarrow \quad y(t) = \underbrace{\pm h_T(t) * h_R(t)}_{\substack{\text{signal component} \\ s(t)}} + \underbrace{w(t) * h_R(t)}_{\substack{\text{noise component} \\ n(t)}}, \quad 0 \le t < T$

(filtered white noise)

* Goal: maximize $\gamma = \dfrac{|s(T)|^2}{E(n^2(t))}$

$|s(T)|^2$ → instantaneous signal power

$E(n^2(t))$ → average noise power (since $n(t)$ is a random process, it does not make much sense to work with $n(T)$.)

* Noise Power

$w(t)$ (white noise) $\rightarrow \boxed{h_R(t)} \rightarrow n(t)$ (filtered noise)

$$S_N(f) = |H_R(f)|^2 \underbrace{S_w(f)}_{N_0/2}$$

$$E\left[n^2(t)\right] = R_N(0) = \int_{-\infty}^{\infty} S_N(f)\,df = \int_{-\infty}^{\infty} \frac{N_0}{2} |H_R(f)|^2\,df$$

* Assume w.l.o.g. that a "1" is transmitted

$$s(T) = h_T(t) * h_R(t)\Big|_{t=T} \quad \text{~ convolution evaluated at } t=T.$$

$$= \mathcal{F}^{-1}\left\{H_T(f) H_R(f)\right\}\Big|_{t=T} = \int_{-\infty}^{\infty} H_T(f) H_R(f) e^{j2\pi fT}\,df$$

* Q: For a given $h_T(t)$, find $h_R(t)$ that will maximize

$$\eta = \frac{\left| \int_{-\infty}^{\infty} H_R(f) H_T(f) e^{j2\pi f T} df \right|^2}{\frac{N_0}{2} \int_{-\infty}^{\infty} |H_R(f)|^2 df}$$

Use Schwarz's Inequality: For finite-energy $\phi_1(x)$ and $\phi_2(x)$,

i.e., $\int_{-\infty}^{\infty} |\phi_i(x)|^2 dx < \infty$

$$\left| \int_{-\infty}^{\infty} \phi_1(x)\phi_2(x) dx \right|^2 \le \int_{-\infty}^{\infty} |\phi_1(x)|^2 dx \int_{-\infty}^{\infty} |\phi_2(x)|^2 dx,$$ with equality if $\phi_1(x) = \underbrace{c}_{\text{constant}} \phi_2^*(x)$

Choose $H_R(f) = \phi_1$ and $H_T(f) e^{j2\pi f T} = \phi_2$

$$\rightarrow \eta = \frac{\left| \int_{-\infty}^{\infty} \phi_1(f)\phi_2(f) df \right|^2}{\frac{N_0}{2} \int_{-\infty}^{\infty} |\phi_1(f)|^2 df} \le \frac{\cancel{\int_{-\infty}^{\infty} |\phi_1(f)|^2 df} \int_{-\infty}^{\infty} |\phi_2(f)|^2 df}{\frac{N_0}{2} \cancel{\int_{-\infty}^{\infty} |\phi_1(f)|^2 df}} = \frac{2}{N_0} \int_{-\infty}^{\infty} |H_T(f)|^2 df$$

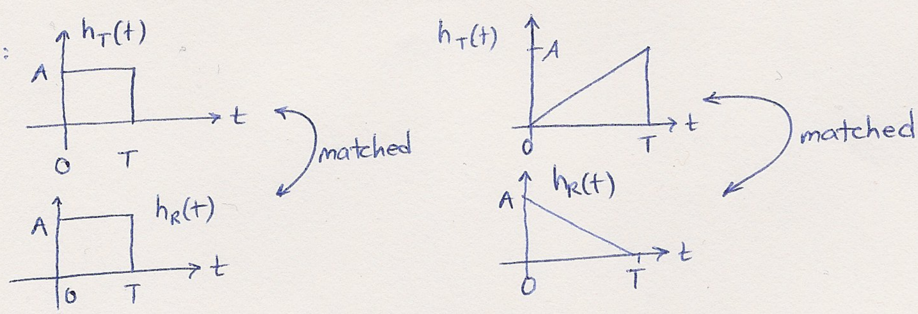
$$\rightarrow \eta_{max} = \frac{2}{N_0} \int_{-\infty}^{\infty} |H_T(f)|^2 df$$

achieved when $H_{R,opt}(f) = c H_T^*(f) e^{-j2\pi f T} \equiv \boxed{h_{R,opt}(t) = c\, h_{T,opt}(T-t)}$

c → constant

w.l.o.g, assume $c = 1$

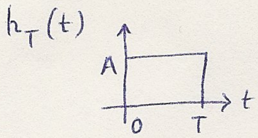
Ex:



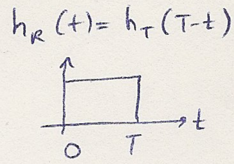
MF pairs:

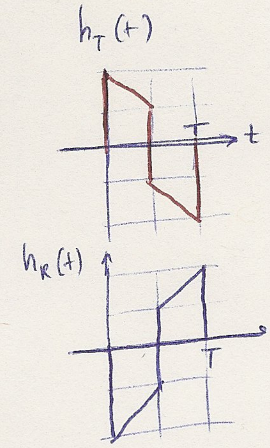
$$\boxed{\begin{array}{l} h_T(t) \rightarrow h_R(t) = h(T-t) \\ |H_T(f)| \rightarrow |H_R(f)| = |H_T(f)| \\ \int_{-\infty}^{\infty} |h_T(t)|^2 dt = \int_{-\infty}^{\infty} |h_R(t)|^2 dt = \text{signal energy} \end{array}}$$

$$\rightarrow \eta_{max} = \frac{2E}{N_0}$$

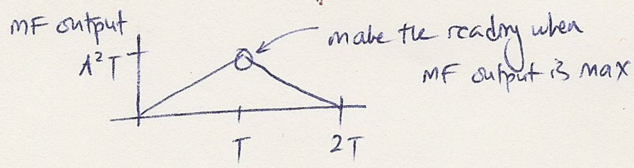


$\rightarrow$

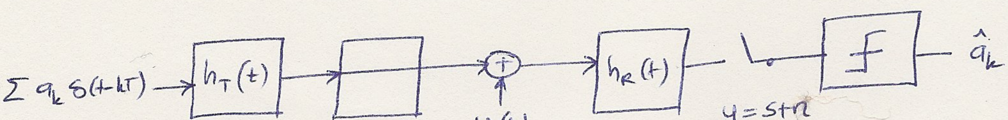




η : does not depend on pulse shape.
It depends only on pulse energy!



P_e analysis



$y = s + n$

$n \rightarrow G(0;\sigma^2)$, $\sigma^2 = \frac{N_0}{2}E$

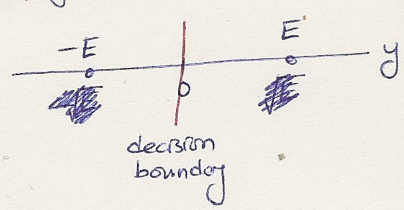
$s \rightarrow \pm E$

$y : G(\pm E;\ \sigma^2 = \frac{N_0}{2}E)$

optimum decision : MAP $\xrightarrow[\text{equally likely}]{\text{if}}$ MAP = ML (= MD)

Assume any antipodal pulse

Constellation



$$y = \begin{cases} E+n, & a_k = 1 \\ -E+n, & a_k = -1 \end{cases}$$

$$P_e = P(\hat{a} \neq a) = P_{e|1}P_1 + P_{e|-1}P_{-1} = \frac{1}{2}P_{e|1} + \frac{1}{2}P_{e|-1}$$

$P_{e|1} = P(\hat{a} \neq a \mid a = 1)$

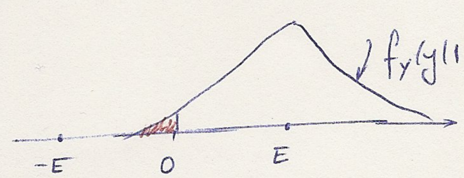
$$P_{e|1} = P(\hat{a} \neq a \mid a=1) = P(\hat{a} = -1 \mid a = 1) = P(y \leq 0 \mid y = E+n)$$

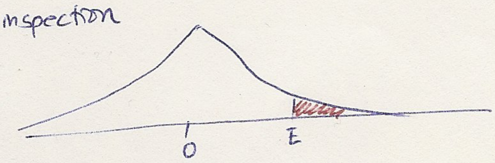
important

$$= \int_{-\infty}^{0} f_Y(y|1)\,dy$$

by inspection

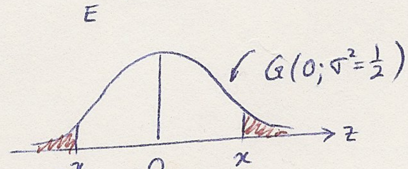
$$= \int_{E}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2\sigma^2}}\,dy$$





complementary error function

Define $\mathrm{erfc}(x) \triangleq \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-z^2}\,dz$



$$P_{e|1} = \int_E^\infty \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2\sigma^2}}\,dy = \int_{\frac{E}{\sqrt{2}\sigma}}^{\infty} \frac{1}{\sqrt{\pi}} e^{-z^2}\,dz = \frac{1}{2}\mathrm{erfc}\left(\frac{E}{\sqrt{2}\sigma}\right) = \frac{1}{2}\mathrm{erfc}\left(\frac{E}{\sqrt{2}\sqrt{\frac{N_0}{2}E}}\right) = \frac{1}{2}\mathrm{erfc}\left(\sqrt{\frac{E}{N_0}}\right)$$

$\frac{y^2}{2\sigma^2} = z^2$

$y = \sqrt{2}\sigma z$

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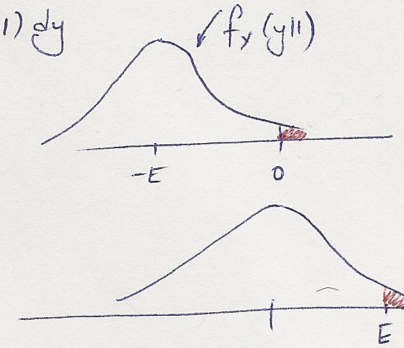
$$P_{e|-1} = P(\hat{a}=1 \mid a=-1) = P(y \geq 0 \mid y = -E+n) = \int_0^\infty f_y(y|-1)\,dy$$



by inspection: $P_{e|-1} = P_{e|1}$

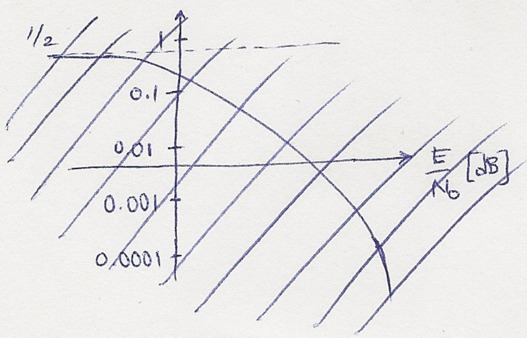
$$\rightarrow \quad P_e = \frac{1}{2}P_{e|1} + \frac{1}{2}P_{e|-1}$$

$$\boxed{P_e = \frac{1}{2}\operatorname{erfc}\sqrt{\frac{E}{N_0}}}$$

$\frac{E}{N_0} \geq 0$

$\frac{E}{N_0} = 0 \Rightarrow \frac{1}{2}\operatorname{erfc}\left(\frac{E}{\sqrt{N_0}}\right) = \frac{1}{2}$

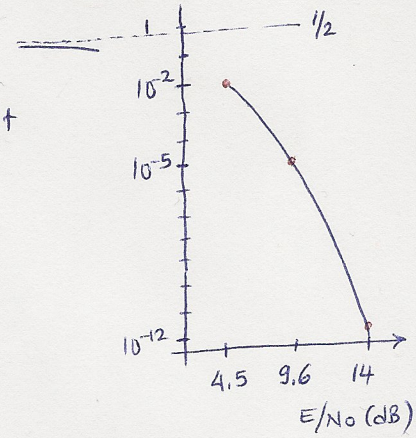
$\frac{E}{N_0} \to \infty \Rightarrow \frac{1}{2}\operatorname{erfc}\left(\frac{E}{\sqrt{N_0}}\right) \to 0$



[Fig 4.6] exponential improvement

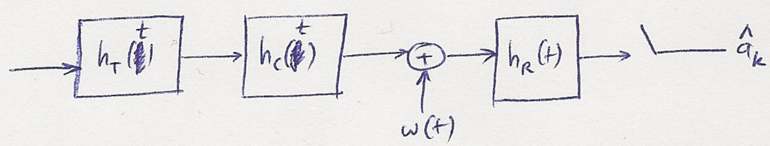
$$\operatorname{erfc}(x) \leq \frac{e^{-x^2}}{\sqrt{\pi}\,x}$$

$$P_e = \frac{1}{2}\operatorname{erfc}\sqrt{\frac{E}{N_0}} < \frac{e^{-E/N_0}}{2\sqrt{\pi}\sqrt{\frac{E}{N_0}}}$$



Binary

Intersymbol Interference ~~([illegible])~~



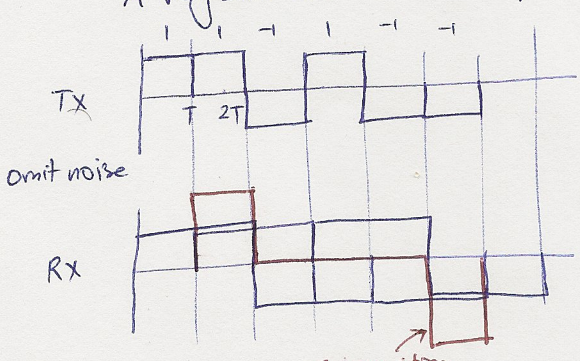
Ex: two-path channel

$h_c(t) = \alpha\delta(t) + \beta\delta(t-\Delta t)$

if $\Delta t = T$ (symbol time) $\rightarrow h_c(t) = \alpha\delta(t) + \beta\delta(t-T)$

[Ex: $R = 100$ kbps, $T = 10^{-5}$
$\Delta t = \frac{\Delta d}{c}$ $\quad \Delta d = 3000$ m]

A very bad scenario: $\alpha = \beta$



$$P_e = P(\hat{a}_k \neq a_k) = P(\hat{a}_k \neq a_k \mid a_k = a_{k-1})P(a_k = a_{k-1}) + P(\hat{a}_k \neq a_k \mid a_k \neq a_{k-1})P(a_k \neq a_{k-1})$$

$$P_e = \frac{1}{2}\underbrace{P(\hat{a}_k \neq a_k \mid a_k = a_{k-1})}_{0} + \frac{1}{2}\underbrace{P(\hat{a}_k \neq a_k \mid a_k \neq a_{k-1})}_{1/2}$$

$$P_e = \frac{1}{2}P(\hat{a}_k \neq a_k \mid y = \pm 2A) + \frac{1}{2}P(\hat{a}_k \neq a_k \mid y = 0) = \frac{1}{4}$$

disjoint events

$P($"$a_k = 1, a_{k-1} = -1$" or "$a_k = -1, a_{k-1} = 1$"$)$

$= P(a_k = 1, a_{k-1} = -1) + P(a_k = -1, a_{k-1} = 1)$

$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$