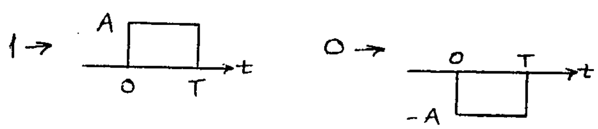
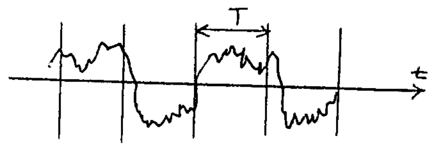
Ch.7 BASEBAND PULSE TRANSMISSION

Some Insight to Detection Theory

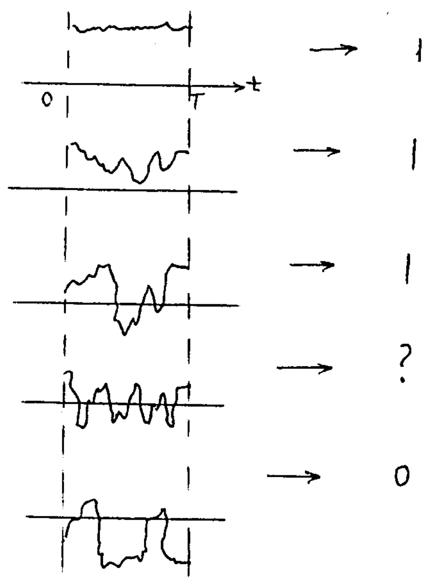


Transmitted pulses



received signal
= transmitted signal
+ noise

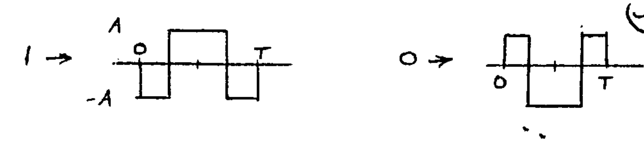
How do you decide whether a "1" or a "0" is transmitted?

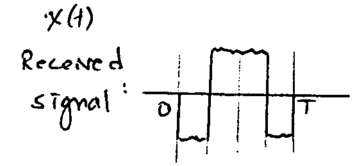


Intuitive detection rule : Integrate the received waveform over a period of bit duration (T)

$$\overset{1}{\underset{0}{\gtrless}} 0$$

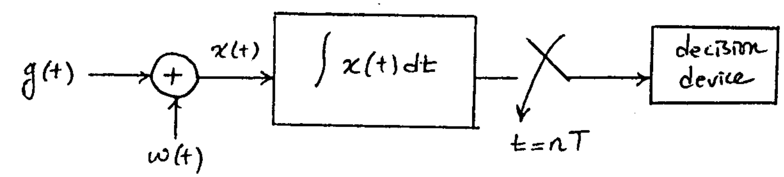
What if



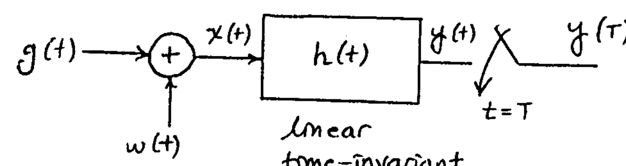


. But if integrate → $\int_0^T x(t) \simeq 0$

∴ Integrator detector is not a general structure



Matched Filter Detector



- white
- stationary
- zero mean
- PSD: $N_0/2$

maximize the signal-to-noise ratio (SNR) at the output

$$x(t) = g(t) + w(t)$$

$$y(t) = x(t) * h(t) = \big(g(t) + w(t)\big) * h(t)$$

$$= g(t) * h(t) + w(t) * h(t)$$

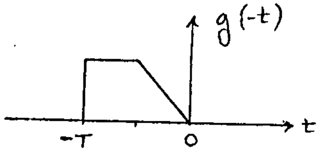
$$= g_0(t) + n(t)$$

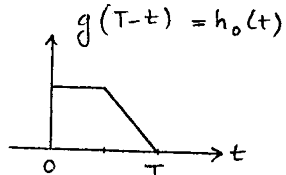
$$SNR\Big|_{output} = \frac{\text{instantaneous signal power}}{\text{average power of noise}} \leftarrow \text{measured at } t=T$$

$$\eta = \frac{|g_o(T)|^2}{E[n^2(t)]}$$

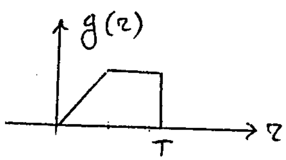
Choosing $\underbrace{h_o(t) = g(T-t)}_{\text{Matched Filter Receiver}}$ maximizes η

Find $g(t) * h_o(t) = g_o(t)$

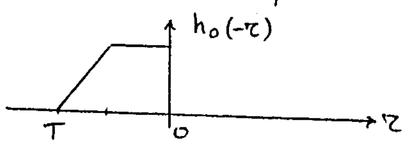




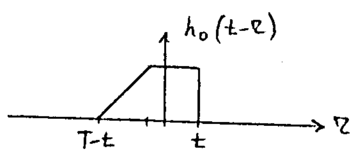
— lec (16), Oct 18

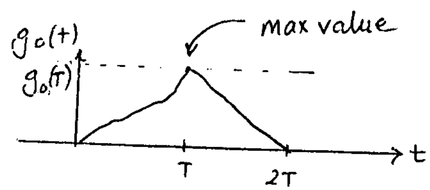


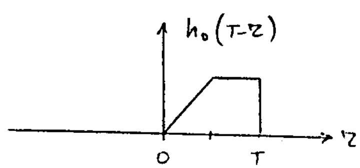
$$g_o(t) = g(t) * h_o(t) = \int_{-\infty}^{\infty} g(\tau) h_o(t-\tau)\, d\tau$$



$$g_o(T) = \int_{-\infty}^{\infty} g(\tau) h_o(T-\tau)\, d\tau$$

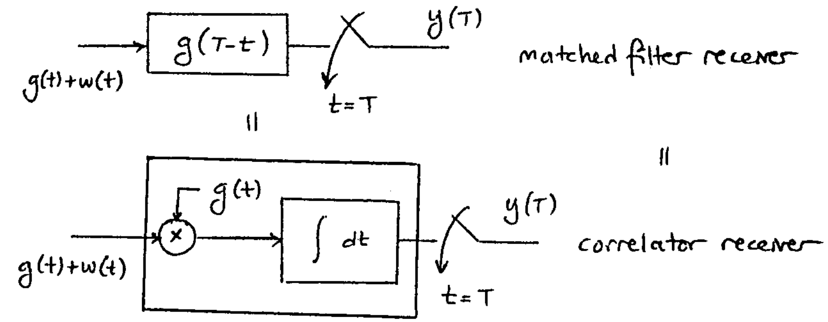






$$g_o(T) = \int_{-\infty}^{\infty} g^2(\tau)\, d\tau \; : \text{energy} = E$$

matched filter = correlator receiver



What is η_{max}? (η_{max} occurs when a matched filter is used)

$$\eta_{max} = \frac{|g_o(T)|^2}{E[n^2(t)]} \qquad g_o(T) = E$$

$$E[n^2(t)] = \int_{-\infty}^{\infty} S_N(f)\, df$$

$$n(t) = w(t) * h(t)$$

$$S_N(f) = |H(f)|^2 S_w(f) = \frac{N_0}{2} |H(f)|^2$$

$$E[n^2(t)] = \frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df$$

$$= \frac{N_0}{2} \int_{-\infty}^{\infty} |h(t)|^2 dt = \frac{N_0}{2} E$$

$$\therefore \eta_{max} = \frac{2E}{N_0}$$

In evaluating the ability of a MF receiver to combat AWGN, we find that all signals that have the same energy are equally effective.

Effect of Amplification

$$g(t)+w(t) \longrightarrow \boxed{h'(t) = k\,h(t)} \longrightarrow y'(t)$$

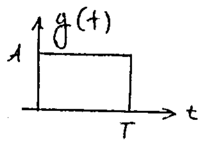
$$y'(t) = \underbrace{g(t) * h'(t)}_{g_o'(t)} + \underbrace{w(t) * h'(t)}_{n'(t)}$$

$$= k\,\underbrace{g(t)*h(t)}_{g_o(t)} + k\,\underbrace{w(t)*h(t)}_{n(t)}$$

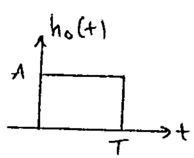
$$\eta' = \frac{|g_o'(T)|^2}{E[n'^2(t)]} = \frac{k^2|g_o(T)|^2}{E[k^2 n^2(t)]} = \eta$$

$\therefore$ Amplification does not affect the SNR!

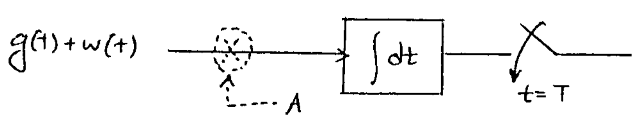
Ex:



$h_o(t) = g(T-t)$



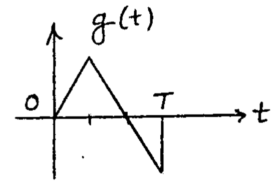
Correlator Receiver Implementation



(the intuitive structure in p-31)

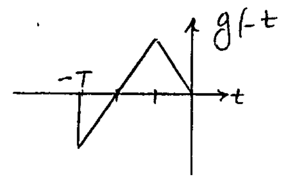
Integrate-and-dump circuit
(MF for NRZ signalling)

Ex:

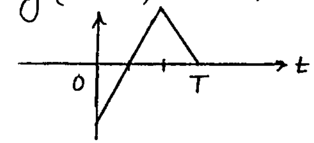


Find the MF $h_o(t)$.

$h_o(t) \triangleq g(T-t)$

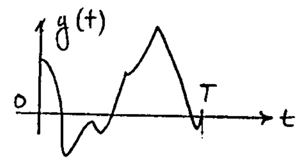


$g(-t+T) = h_o(t)$

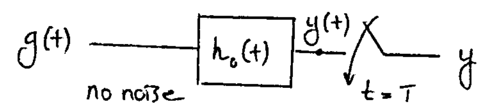


Check: $h_o(t-\tau)\Big|_{t=T} \overset{?}{=} g(t)$ ✓

Ex:



What is the MF output if there is no noise?

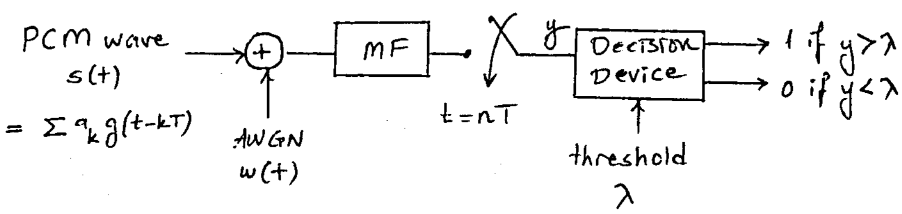


$y = E_g$ (energy)

More Insight on MF

- Since amplification does not affect SNR, what is important is the <u>shape</u> of $h(t)$.
- Then, what shape to choose?
- Since noise is stationary, its average power is the same at all t.
- Therefore, emphasize the portion of $g(t)$ that is more robust against noise.

Receiver Structure



Some Insight on Threshold Selection

(a) "0"s and "1"s : equally-likely $\rightarrow$ $P_0 = P_1 = \frac{1}{2}$

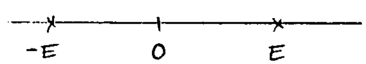
(i) Antipodal signalling

$1 \rightarrow g(t)$

$0 \rightarrow -g(t)$

$$\Rightarrow \quad y = \pm E + \underbrace{\underbrace{\left(h_0(t) * w(t)\right)\Big|_{t=T}}_{n(T)}}_{n}$$

$$= \pm E + n$$



MF output when there is no noise and a "0" is tx'ed

MF output when there is no noise and a "1" is tx'ed.

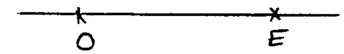
Intuitive threshold $\rightarrow$ $\lambda = 0$

— lec 17, Oct 20

(ii) Non-antipodal signalling

- Special case : OOK

$$y = \begin{cases} E + n, & \text{if } 1 \text{ tx'ed} \\ n, & \text{if } 0 \text{ tx'ed} \end{cases}$$

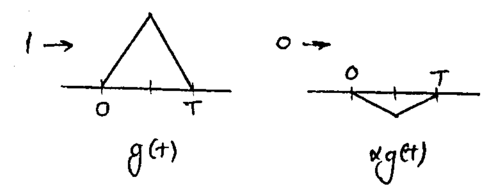


Intuitive threshold $\rightarrow$ $\lambda = \frac{E}{2}$

- general case

$1 \rightarrow g(t)$

$0 \rightarrow \alpha g(t)$



$$\lambda = \frac{\alpha + 1}{2} E$$

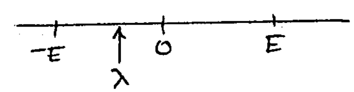
$h_0(t) = g(T - t)$

<u>Rule</u> : For equally-likely signalling, the optimum threshold value (λ_0) is the arithmetic mean of the two possible outcomes of the MF when there is no noise.

(b) Non-equally-likely case

Assume $P_1 \gg P_0$

antipodal $\rightarrow$



limiting case $P_1 = 1$, $P_0 = 0$ $\rightarrow$ $\lambda = -\infty$

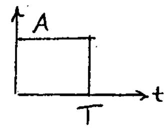
Probability of Error Calculation

$$P_e = \underbrace{P(0|1)P_1}_{\text{prob of sending a "1" but deciding for a "0"}} + \underbrace{P(1|0)P_0}_{P_F\ (\text{false alarm})}$$

P_M (miss)

Assumptions

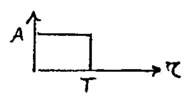
- perfect channel
- AWGN with $N_0/2$ and zero-mean
- MF receiver
- antipodal signalling with $g(t)$



$E_b = A^2T$ (bit energy)

$$y(\text{given } 0) = -A^2T + n$$

$$n = w(t) * h_o(t)\Big|_{t=T} = \int_{-\infty}^{\infty} w(\tau)\underbrace{h(T-\tau)}\,d\tau$$

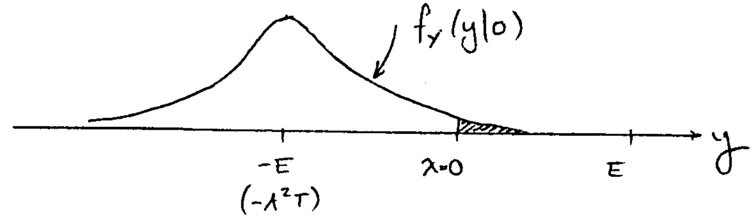


$$= A\int_0^T w(t)\,dt$$

$w(t)$: Gaussian process $\longrightarrow$ $n(t)$: Gaussian process $\longrightarrow$ n: Gaussian rv

$$\mu_n = H(0)\mu_w$$
$$= 0$$

- the decision variable y is a rv.
- If we know the mean and variance of y, we can readily obtain its pdf.



$$P_F = P(1|0) = \int_{-\infty}^{0} f_Y(y|0)\,dy$$

$$y = \underbrace{-A^2T}_{\text{constant}} + n \longrightarrow G(0;\sigma_n^2)$$

$$\rightarrow y: G(-A^2T;\sigma_n^2)$$

$$\sigma_n^2 = E[n_1 n_2] = E\left[A^2\int_0^T\int_0^T w(t)w(u)\,dt\,du\right]$$

$$= A^2\int_0^T\int_0^T E[w(t)w(u)]\,dt\,du$$

$$= A^2\int_0^T\int_0^T R_w(t,u)\,dt\,du$$

$$= A^2\int_0^T\left(\frac{N_0}{2}\underbrace{\int_0^T \delta(t-u)\,dt}_{1}\right)du$$

$$= \frac{A^2N_0}{2}\int_0^T du = \frac{A^2TN_0}{2}$$

$$\rightarrow f_Y(y|0) = \frac{1}{\sqrt{2\pi \frac{A^2TN_0}{2}}} e^{-\frac{(y+A^2T)^2}{2\frac{A^2TN_0}{2}}}$$

$$P_F = P(1|0) = \int_0^\infty \frac{1}{\sqrt{\pi A^2TN_0}} e^{-\frac{(y+A^2T)^2}{A^2TN_0}} dy$$

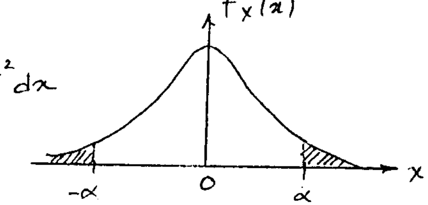
There is no closed form expression for the above integral.

Define a new function $\operatorname{erfc}(\alpha)$ ... complementary error function

$\operatorname{erfc}(\alpha)$ = area under the tails of a Gaussian r.v. with $\mu = 0$ and $\sigma^2 = 1/2$.

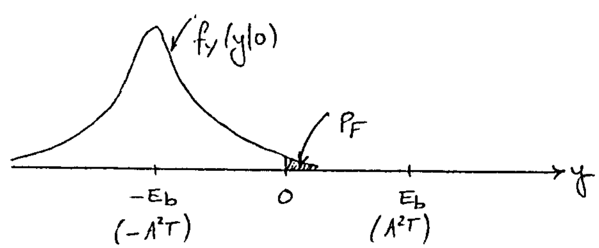
$$f_X(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$$

$$\operatorname{erfc}(\alpha) = \frac{2}{\sqrt{\pi}} \int_\alpha^\infty e^{-x^2} dx$$



$\operatorname{erfc}(\alpha)$ = sum of the shaded areas

Our case:



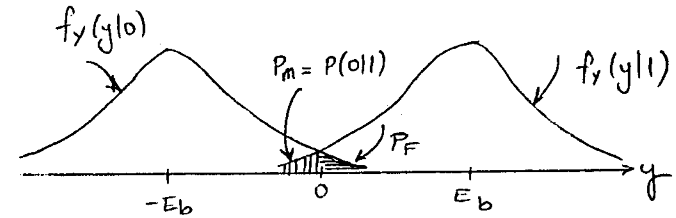
define $u = \dfrac{y + A^2T}{\sqrt{A^2TN_0}}$ $\qquad du = dy \Big/ \sqrt{A^2TN_0}$

$$P_F = \int_{\sqrt{\frac{A^2T}{N_0}}}^\infty \frac{1}{\sqrt{\pi A^2TN_0}} e^{-u^2} \sqrt{A^2TN_0}\, du$$

$$P_F = \frac{1}{\sqrt{\pi}} \int_{\sqrt{\frac{A^2T}{N_0}}}^\infty e^{-u^2} du = \frac{1}{2} \operatorname{erfc} \sqrt{\frac{A^2T}{N_0}}$$

$$= \frac{1}{2} \operatorname{erfc} \sqrt{\frac{E_b}{N_0}}$$

Note that $P_e = P(0|1)P_1 + P(1|0)P_0$



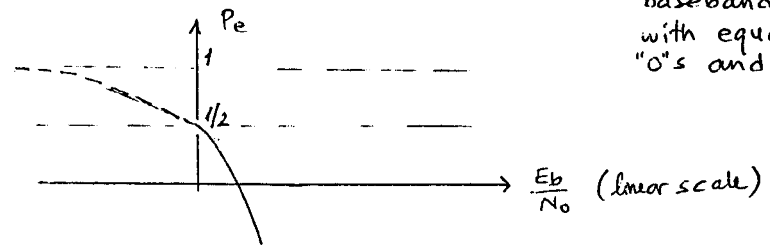
$$y(\text{given } 1) = A^2T + n$$

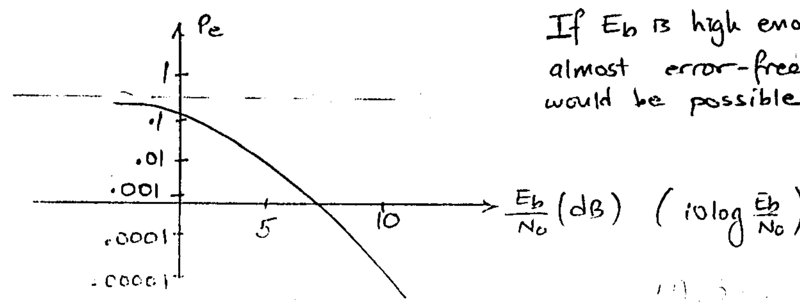
By inspection: $P_m = P_F = \frac{1}{2} \operatorname{erfc} \sqrt{\frac{E_b}{N_0}}$

$$P_e = \frac{1}{2} P_m + \frac{1}{2} P_F$$

$$\rightarrow \boxed{P_e = \frac{1}{2} \operatorname{erfc} \sqrt{\frac{E_b}{N_0}}}$$

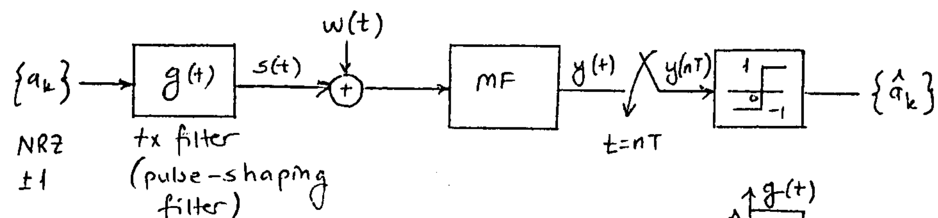
performa.. of MF receiver with any antipodal baseband scheme with equally-likely "0"s and "1"s.

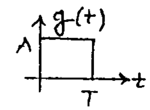




If E_b is high enough, almost error-free transmission would be possible

Mini Review (Noise Performance)





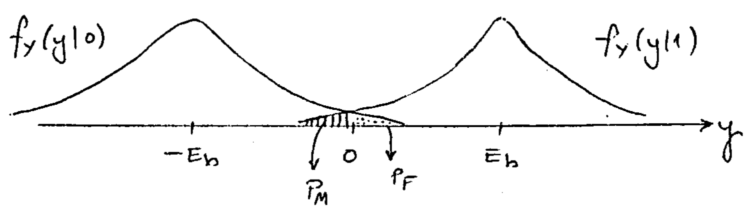
$$P_e = P(\hat{a}_k \neq a_k) = P(0|1)P_1 + P(1|0)P_0$$

$$y = \pm E_b + n$$

$n \rightarrow G(0; \sigma_n^2)$

$y \rightarrow G(\pm E_b; \sigma_n^2)$

Find the pdfs of $y|0$ and $y|1$, and compute P_M and P_F.

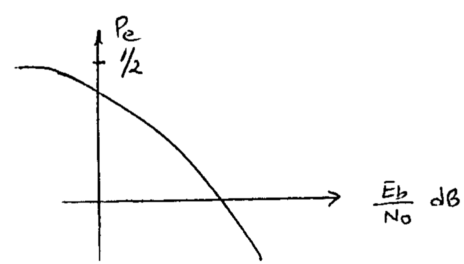


$$P_F = \int_0^\infty \frac{1}{\sqrt{2\pi\sigma_n^2}} e^{-\frac{(y+E_b)^2}{2\sigma_n^2}} dy$$

$$\frac{y+E_b}{\sqrt{2}\sigma_n} = u \quad \longrightarrow \quad P_F = \frac{1}{\sqrt{\pi}} \int_{E_b/\sqrt{2}\sigma_n}^{\infty} e^{-u^2} du = \frac{1}{2} \operatorname{erfc}\left(\frac{E_b}{\sqrt{2}\sigma_n}\right)$$

$$\sigma_n^2 = \frac{N_0}{2} E_b$$

$$P_e = \frac{1}{2} \operatorname{erfc}\sqrt{\frac{E_b}{N_0}}$$

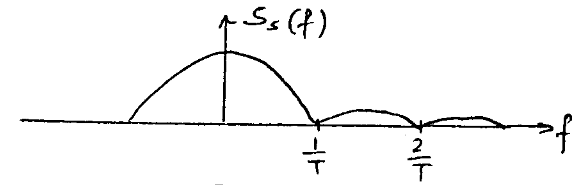


Observations

1) For NRZ, instead of the matched filter, we could have used an integrate-and-dump cct; performance should be the same (CHECK!)

2) Performance depends on E_b/N_0; all pulse shapes that have the same energy will yield the same performance.

3) Note that $S_s(f) \propto |G(f)|^2$



Average Signal Power $= \frac{E_b}{T} = A^2$

Signal BW $\simeq \frac{1}{T}$

Noise Power in signal BW $= \frac{N_0}{2} \times 2 \times \frac{1}{T} = \frac{N_0}{T}$

$$\frac{\text{signal power}}{\text{noise power in signal BW}} = \frac{A^2}{N_0/T} = \frac{A^2 T}{N_0} = \frac{E_b}{N_0} = SNR$$

4) Performance improves if
   a) E_b is increased
   b) bit rate is decreased (for a given power)

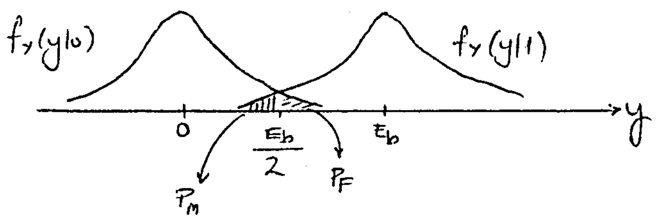
5) Performance improves drastically with a moderate increase in E_b. If the quantization error is unnoticable, the original analog signal can be reconstructed by almost no error. (Regenerative repeaters)

Note that this cannot be said for analog communications.

Noise Performance for OOK (Prob. 7.4)

- Same structure with $a_k \in \{0,1\}$

$$y = \begin{cases} E_b + n \\ n \end{cases}$$



$$P_F = \int_{E/2}^{\infty} \frac{1}{\sqrt{2\pi\sigma_n^2}} e^{-\frac{y^2}{2\sigma_n^2}} dy$$

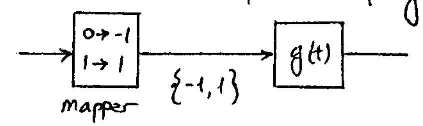
$$u = \frac{y}{\sqrt{2}\sigma_n} \rightarrow P_F = \int_{\frac{E_b}{2\sqrt{2}\sigma_n}}^{\infty} \frac{1}{\sqrt{\pi}} e^{-u^2} du$$

$$P_e = \frac{1}{2} \operatorname{erfc}\left(\frac{E_b}{2\sqrt{2}\sigma_n}\right) = \frac{1}{2} \operatorname{erfc}\left(\frac{1}{2}\sqrt{\frac{E_b}{N_0}}\right)$$

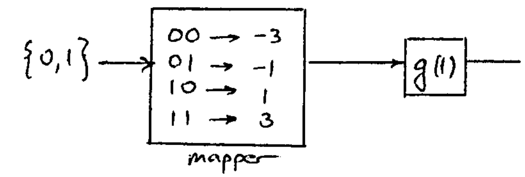
In order that OOK and NRZ schemes have the same error performance, $E_{b,OOK} = 4 E_{b,NRZ}$

M-ary PAM

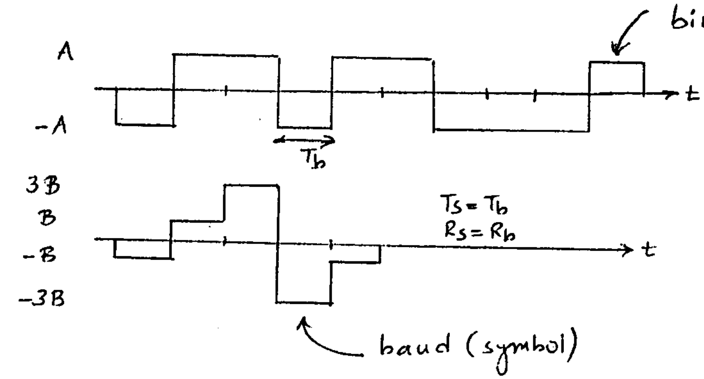
binary PAM: $\{0,1\}$



quaternary PAM $(M=4)$ $\{0,1\}$

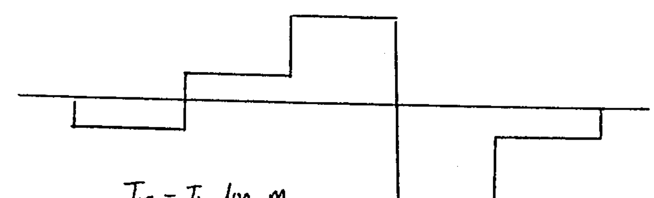


0 1 1 0 1 1 0 0 0 1



$\log_2 M$ times faster (same BW)

1 baud = $\log_2 M$ bits



$\log_2 M$ times less BW (same bitrate)

$$T_S = T_b \log_2 M$$

$$R_S = R_b \frac{1}{\log_2 M}$$

In order to have the same performance as binary PAM, the transmit power must be increased by a factor of $M^2/\log_2 M$ in an M-ary PAM scheme.