Carleton University

Department of Systems and Computer Engineering

SYSC 4600 – Digital Communications – Fall 2009

Assignment #3

Posted on Wednesday, November 4th, 2009

Due on Friday, November 13th at 4 pm in the assignment box.

Marking Scheme: Q1[25%], Q2[25%], Q3[50% + 5% bonus for constellation (e)].

Question 1

A communication system uses two signals $s_1(t) = a\phi(t)$ and $s_2(t) = (a+1)\phi(t)$ to transmit binary one and zero bits respectively, each equally likely, where a is a real number, and $\phi(t)$ is a bandpass signal with energy 1. The communication is corrupted by AWGN with PSD $N_0/2$ and received via a coherent receiver with filter matched to $\phi(t)$.

1. Assuming ASK transmission:
   (a) Find a.
   (b) What is the average energy per bit $\mathcal{E}_\mathrm{b}$?
   (c) What is the probability of bit error P_e as a function of $\mathcal{E}_\mathrm{b}/N_0$? You may use formulas from class.
2. Answer the same questions for PSK transmission.
3. Now for any a:
   (a) Find P_e as a function of $\mathcal{E}_\mathrm{b}/N_0$ and a.
   (b) What is the choice of a that gives the smallest P_e for a given $\mathcal{E}_\mathrm{b}/N_0$? (Proof required)

Solution

1. Assuming ASK transmission,
   (a) $a = 0$ (or $a = -1$)
   (b) $\mathcal{E}_\mathrm{b} = \frac{1}{2}\mathcal{E}_0 + \frac{1}{2}\mathcal{E}_1 = \frac{1}{2}$
   (c) $d = 1$, $P_\mathrm{e} = \frac{1}{2}\operatorname{erfc}\left(\frac{d}{2\sqrt{N_0}}\right) = \frac{1}{2}\operatorname{erfc}\left(\frac{1}{2/\sqrt{N_0}}\right) = \frac{1}{2}\operatorname{erfc}\left(\frac{1}{\sqrt{2}}\sqrt{\frac{\mathcal{E}_\mathrm{b}}{N_0}}\right)$.

2. Assuming PSK transmission,
   (a) $a = -\frac{1}{2}$
   (b) $\mathcal{E}_\mathrm{b} = \frac{1}{2}\mathcal{E}_0 + \frac{1}{2}\mathcal{E}_1 = \frac{1}{2}\frac{1}{4} + \frac{1}{2}\frac{1}{4} = \frac{1}{4}$
   (c) $d = 1$, $P_\mathrm{e} = \frac{1}{2}\,\mathrm{erfc}\left(\frac{d}{2\sqrt{N_0}}\right) = \frac{1}{2}\,\mathrm{erfc}\left(\frac{1}{2/\sqrt{N_0}}\right) = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{\mathcal{E}_\mathrm{b}}{N_0}}\right)$.

3. Now for any a:
   (a) We have $P_\mathrm{e} = \frac{1}{2}\,\mathrm{erfc}\left(\frac{1}{2/\sqrt{N_0}}\right)$. Now $\mathcal{E}_\mathrm{b} = \frac{1}{2}\mathcal{E}_0 + \frac{1}{2}\mathcal{E}_1 = a^2 + a + \frac{1}{2}$. We can therefore write $P_\mathrm{e} = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{\mathcal{E}_\mathrm{b}}{N_0}}\frac{1}{2\sqrt{a^2+a+\frac{1}{2}}}\right)$
   (b) The function erfc is monotone decreasing, and inversion is also a monotone decreasing for positive numbers, while the square root is monotone increasing, therefore P_e grows with $a^2+a+\frac{1}{2}$. Therefore P_e is minimised when $a^2+a+\frac{1}{2}$ is minimised. Now $a^2+a+\frac{1}{2}$ is an upward parabola with minimum at $\frac{\partial}{\partial a}\left(a^2+a+\frac{1}{2}\right) = 2a+1 = 0 \Rightarrow a = -\frac{1}{2}$. Therefore PSK is the most energy–efficient transmission scheme.

Question 2

Consider a 4FSK system which uses four frequencies $f_1 = 1$ GHz, $f_2 = 1.02$ GHz, $f_3 = 1.05$ GHz, $f_4 = 1.09$ GHz. The four symbols are therefore $s_i(t) = A\cos(2\pi f_i t)$, $0 \le t \le T_\mathrm{S}$. Assume $A = 1$.

1. Assuming $T_\mathrm{S} = 25$ ns, are the four signals orthogonal?
2. Find the lowest T_S that ensures all four signals are orthogonal. What is the corresponding transmission bit rate?

Hint: For both questions, it is best to find $\langle s_i, s_j\rangle$ for general f_i, f_j.

Solution

Using the hint, we find

$$\begin{aligned}\langle s_i, s_j\rangle &= \int_0^{T_\mathrm{S}} \cos(2\pi f_i t)\cos(2\pi f_j t)dt = \frac{1}{2}\int_0^{T_\mathrm{S}} \cos(2\pi(f_i+f_j)t)dt + \frac{1}{2}\int_0^{T_\mathrm{S}} \cos(2\pi(f_i-f_j)t)dt \\ &= \frac{-\sin(2\pi(f_i+f_j)T_\mathrm{S})}{4\pi(f_i+f_j)} + \frac{-\sin(2\pi(f_i-f_j)T_\mathrm{S})}{4\pi(f_i-f_j)} \approx \frac{-\sin(2\pi(f_i-f_j)T_\mathrm{S})}{4\pi(f_i-f_j)}\end{aligned}$$

Then

1. For $T_\mathrm{S} = 25$ ns,
$$\langle s_2, s_3\rangle \approx \frac{-\sin(2\pi 0.0325)}{4\pi 0.03\ \text{GHz}} = \frac{-\sin(1.5\pi)}{4\pi 0.03\ \text{GHz}} \neq 0.$$
   Since two of the signals are not orthogonal, the four signals are not orthogonal.

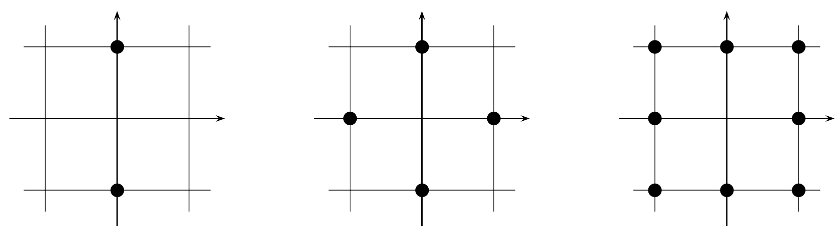
2. To have all signals orthogonal, we need all inner products zero for $i \neq j$. We need:

$$\begin{cases} \langle s_1, s_2 \rangle \approx 0 \Rightarrow 0.02 \text{ GHz} \times T_\text{S} = \frac{1}{2}k_1, \\ \langle s_1, s_3 \rangle \approx 0 \Rightarrow 0.05 \text{ GHz} \times T_\text{S} = \frac{1}{2}k_2, \\ \langle s_1, s_4 \rangle \approx 0 \Rightarrow 0.09 \text{ GHz} \times T_\text{S} = \frac{1}{2}k_3, \\ \langle s_2, s_3 \rangle \approx 0 \Rightarrow 0.03 \text{ GHz} \times T_\text{S} = \frac{1}{2}k_4, \\ \langle s_2, s_4 \rangle \approx 0 \Rightarrow 0.07 \text{ GHz} \times T_\text{S} = \frac{1}{2}k_5, \\ \langle s_3, s_4 \rangle \approx 0 \Rightarrow 0.04 \text{ GHz} \times T_\text{S} = \frac{1}{2}k_6, \end{cases} \Rightarrow \begin{cases} T_\text{S} = k_1 \times 25 \text{ ns}, \\ T_\text{S} = k_2 \times 10 \text{ ns}, \\ T_\text{S} = k_3 \times \frac{50}{9} \text{ ns}, \\ T_\text{S} = k_4 \times \frac{50}{3} \text{ ns}, \\ T_\text{S} = k_5 \times \frac{50}{7} \text{ ns}, \\ T_\text{S} = k_6 \times 12.5 \text{ ns}, \end{cases}$$

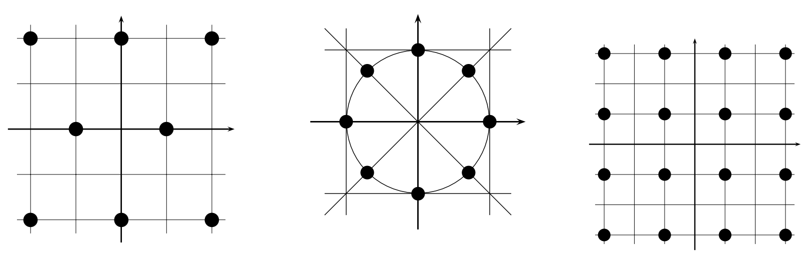
with all k_i integers. The smallest T_S that verifies this is $T_\text{S} = 50$ ns. This corresponds to 20M symbols/second, i.e., 40Mbps.

Question 3

Consider the 6 following signal constellations, with each division representing one unit:



(a) BPSK (b) QPSK (c) 8QAM



(d) 8QAM–Mod (e) 8PSK (bonus) (f) 16QAM

(Note: you may get full marks if you ignore constellation (e), which is worth bonus marks)

1. For each of the 5(6) constellations, find:

(a) The average energy per symbol $\mathcal{E}_\mathrm{S}$ and the average energy per bit $\mathcal{E}_\mathrm{b}$.

(b) Assuming that the probability of error is dominated by errors between the nearest points in the constellation, find $d_{\min}$ the smallest distance between two points. Then find the probability that a given symbol is received with error, using the approximation

$$P_\mathrm{e}(\text{symbol}) \approx \tfrac{1}{2}\,\mathrm{erfc}\left(\frac{d_{\min}}{2\sqrt{N_0}}\right)$$

to find $P_\mathrm{e}(\text{symbol})$ in the form $\frac{1}{2}\,\mathrm{erfc}\left(\alpha\sqrt{\frac{\mathcal{E}_\mathrm{b}}{N_0}}\right)$ (basically, you must find the constant α for every constellation).

2. If we want to achieve a particular $P_\mathrm{e}(\text{symbol})$, sort the constellations from most energy-efficient to least. Justify your answer.

3. Now assume that the symbols are grey–coded, meaning that (practically) every symbol error results in one bit error only. For each of the 5(6) constellations:

   (a) Find the (approximate) probability $P_\mathrm{e}(\text{bit})$ that a given received bit is in error, as a function of $\mathcal{E}_\mathrm{b}/N_0$.

   (b) Assuming $\mathcal{E}_\mathrm{b}/N_0 = 13\,\mathrm{dB}$, find the numeric value of $P_\mathrm{e}(\text{bit})$ (hint: use the *MATLAB* function *erfc*).

4. When $\mathcal{E}_\mathrm{b}/N_0 = 13\,\mathrm{dB}$, sort the constellations from lowest $P_\mathrm{e}(\text{bit})$ to highest.

Solution

1. The parameters are:

System	BPSK	QPSK	8QAM	8QAM–Mod	8PSK	16QAM
bits per symbol	1	2	3	3	3	4
$\mathcal{E}_\mathrm{S}$	1	1	$\frac{3}{2}$	$\frac{21}{4}$	1	10
$\mathcal{E}_\mathrm{b}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{7}{4}$	$\frac{1}{3}$	$\frac{5}{2}$
$d_{\min}$	2	$\sqrt{2}$	1	2	$2\sin\frac{\pi}{8}$	2
α	1	1	$\frac{1}{\sqrt{2}}$ ≈ 0.7071	$\sqrt{\frac{4}{7}}$ ≈ 0.7559	$\sqrt{3}\sin\frac{\pi}{8}$ ≈ 0.6628	$\sqrt{\frac{2}{5}}$ ≈ 0.6325

We find α like so:

$$\tfrac{1}{2}\,\mathrm{erfc}\left(\frac{d_{\min}}{2\sqrt{N_0}}\right) = \tfrac{1}{2}\,\mathrm{erfc}\left(\alpha\sqrt{\frac{\mathcal{E}_\mathrm{b}}{N_0}}\right) \Rightarrow \alpha = \frac{d_{\min}}{2\sqrt{\mathcal{E}_\mathrm{b}}}.$$

2. The higher the α, the lower the $\mathcal{E}_\mathrm{b}/N_0$ needed to achieve a particular P_e(symbol). Therefore, constellations with higher α are more energy efficient for a given P_e(symbol):
   BPSK = QPSK > 8QAM–Mod > 8QAM > 8PSK > 16QAM.

3. For grey–coded constellations:
   We have $P_\mathrm{e}(\text{bit}) = P_\mathrm{e}(\text{symbol})/(\text{bits per symbol})$:

System	P_e(bit)	P_e(bit) when $\mathcal{E}_\mathrm{b}/N_0 = 13\,\mathrm{dB} \approx 19.95$
BPSK	$\frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{\mathcal{E}_\mathrm{b}}{N_0}}\right)$	$1.333 \cdot 10^{-10}$
QPSK	$\frac{1}{4}\,\mathrm{erfc}\left(\sqrt{\frac{\mathcal{E}_\mathrm{b}}{N_0}}\right)$	$6.665 \cdot 10^{-11}$
8QAM	$\frac{1}{6}\,\mathrm{erfc}\left(\frac{1}{\sqrt{2}}\sqrt{\frac{\mathcal{E}_\mathrm{b}}{N_0}}\right)$	$1.323 \cdot 10^{-6}$
8QAM–Mod	$\frac{1}{6}\,\mathrm{erfc}\left(\sqrt{\frac{4}{7}}\sqrt{\frac{\mathcal{E}_\mathrm{b}}{N_0}}\right)$	$2.991 \cdot 10^{-7}$
8PSK	$\frac{1}{6}\,\mathrm{erfc}\left(\sqrt{3}\sin\frac{\pi}{8}\sqrt{\frac{\mathcal{E}_\mathrm{b}}{N_0}}\right)$	$4.709 \cdot 10^{-6}$
16QAM	$\frac{1}{8}\,\mathrm{erfc}\left(\sqrt{\frac{2}{5}}\sqrt{\frac{\mathcal{E}_\mathrm{b}}{N_0}}\right)$	$8.078 \cdot 10^{-6}$

   (Note: do not forget to convert $\mathcal{E}_\mathrm{b}/N_0$ from dB to linear scale.)

4. QPSK > BPSK > 8QAM–Mod > 8QAM > 8PSK > 16QAM.