Assignment #3

Posted on Wednesday, November 4th, 2009
Due on Friday, November 13th at 4 pm in the assignment box.

Marking Scheme: Q1[25%], Q2[25%], Q3[50% + 5% bonus for constellation (e)].

Question 1
A communication system uses two signals \( s_1(t) = a\phi(t) \) and \( s_2(t) = (a + 1)\phi(t) \) to transmit binary one and zero bits respectively, each equally likely, where \( a \) is a real number, and \( \phi(t) \) is a bandpass signal with energy 1. The communication is corrupted by AWGN with PSD \( N_0/2 \) and received via a coherent receiver with filter matched to \( \phi(t) \).

1. Assuming ASK transmission:
   (a) Find \( a \).
   (b) What is the average energy per bit \( E_b \)?
   (c) What is the probability of bit error \( P_e \) as a function of \( E_b/N_0 \)? You may use formulas from class.

2. Answer the same questions for PSK transmission.

3. Now for any \( a \):
   (a) Find \( P_e \) as a function of \( E_b/N_0 \) and \( a \).
   (b) What is the choice of \( a \) that gives the smallest \( P_e \) for a given \( E_b/N_0 \)? (Proof required)

Solution
1. Assuming ASK transmission,
   (a) \( a = 0 \) (or \( a = -1 \))
   (b) \( E_b = \frac{1}{2}E_0 + \frac{1}{2}E_1 = \frac{1}{2} \)
   (c) \( d = 1, P_e = \frac{1}{2} \text{erfc} \left( \frac{d}{2\sqrt{N_0}} \right) = \frac{1}{2} \text{erfc} \left( \frac{1}{2\sqrt{N_0}} \right) = \frac{1}{2} \text{erfc} \left( \frac{1}{\sqrt{2}} \frac{E_b}{N_0} \right) \).
2. Assuming PSK transmission,
   (a) $a = -\frac{1}{2}$
   (b) $E_b = \frac{1}{2}E_0 + \frac{1}{2}E_1 = \frac{1}{2} + \frac{1}{2} = 1$
   (c) $d = 1$, $P_c = \frac{1}{2} \operatorname{erfc}\left(\frac{d}{\sqrt{2N_0}}\right) = \frac{1}{2} \operatorname{erfc}\left(\frac{1}{2\sqrt{N_0}}\right) = \frac{1}{2} \operatorname{erfc}\left(\frac{\sqrt{1/N_0}}{2}\right).

3. Now for any $a$:
   (a) We have $P_c = \frac{1}{2} \operatorname{erfc}\left(\frac{1}{2\sqrt{N_0}}\right)$. Now $E_b = \frac{1}{2}E_0 + \frac{1}{2}E_1 = a^2 + a + \frac{1}{4}$.

   We can therefore write $P_c = \frac{1}{2} \operatorname{erfc}\left(\frac{\sqrt{E_b}}{2\sqrt{a^2 + a + \frac{1}{4}}}\right)$

   (b) The function $\operatorname{erfc}$ is monotone decreasing, and inversion is also a monotone decreasing for positive numbers, while the square root is monotone increasing, therefore $P_c$ grows with $a^2 + a + \frac{1}{4}$. Therefore $P_c$ is minimised when $a^2 + a + \frac{1}{4}$ is minimised. Now $a^2 + a + \frac{1}{4}$ is an upward parabola with minimum at $\frac{a}{2}(a^2 + a + \frac{1}{4}) = 2a + 1 = 0 \Rightarrow a = -\frac{1}{2}$.

   Therefore PSK is the most energy-efficient transmission scheme.

**Question 2**

Consider a 4FSK system which uses four frequencies $f_1 = 1 \text{ GHz}$, $f_2 = 1.02 \text{ GHz}$, $f_3 = 1.05 \text{ GHz}$, $f_4 = 1.09 \text{ GHz}$. The four symbols are therefore $s_i(t) = A \cos(2\pi f_i t)$, $0 \leq t \leq T_S$. Assume $A = 1$.

1. Assuming $T_S = 25 \text{ ns}$, are the four signals orthogonal?

2. Find the lowest $T_S$ that ensures all four signals are orthogonal. What is the corresponding transmission bit rate?

**Hint:** For both questions, it is best to find $\langle s_i, s_j \rangle$ for general $f_i, f_j$.

**Solution**

Using the hint, we find

$$\langle s_i, s_j \rangle = \int_0^{T_S} \cos(2\pi f_i t) \cos(2\pi f_j t) dt = \frac{1}{2} \int_0^{T_S} \cos(2\pi (f_i + f_j) t) dt + \frac{1}{2} \int_0^{T_S} \cos(2\pi (f_i - f_j) t) dt$$

$$= -\frac{\sin(2\pi (f_i + f_j) T_S)}{4\pi (f_i + f_j)} + \frac{-\sin(2\pi (f_i - f_j) T_S)}{4\pi (f_i - f_j)} \approx -\frac{-\sin(2\pi (f_i - f_j) T_S)}{4\pi (f_i - f_j)}$$

Then

1. For $T_S = 25 \text{ ns}$,

   $$\langle s_2, s_3 \rangle \approx \frac{-\sin(2\pi 0.0325)}{4\pi 0.03 \text{ GHz}} = \frac{-\sin(1.5\pi)}{4\pi 0.03 \text{ GHz}} \neq 0.$$

   Since two of the signals are not orthogonal, the four signals are not orthogonal.
2. To have all signals orthogonal, we need all inner products zero for $i \neq j$. We need:

\[
\begin{align*}
  \langle s_1, s_2 \rangle &\approx 0 \Rightarrow 0.02 \text{ GHz} \times T_S = \frac{1}{2} k_1, \\
  \langle s_1, s_3 \rangle &\approx 0 \Rightarrow 0.05 \text{ GHz} \times T_S = \frac{1}{2} k_2, \\
  \langle s_1, s_4 \rangle &\approx 0 \Rightarrow 0.09 \text{ GHz} \times T_S = \frac{1}{2} k_3, \\
  \langle s_2, s_3 \rangle &\approx 0 \Rightarrow 0.03 \text{ GHz} \times T_S = \frac{1}{2} k_4, \\
  \langle s_2, s_4 \rangle &\approx 0 \Rightarrow 0.07 \text{ GHz} \times T_S = \frac{1}{2} k_5, \\
  \langle s_3, s_4 \rangle &\approx 0 \Rightarrow 0.04 \text{ GHz} \times T_S = \frac{1}{2} k_6,
\end{align*}
\]

with all $k_i$ integers. The smallest $T_S$ that verifies this is $T_S = 50$ ns. This corresponds to 20M symbols/second, i.e., 40Mbps.

**Question 3**

Consider the 6 following signal constellations, with each division representing one unit:

(a) BPSK

(b) QPSK

(c) 8QAM

(d) 8QAM–Mod

(e) 8PSK (bonus)

(f) 16QAM

(Note: you may get full marks if you ignore constellation (e), which is worth bonus marks)

1. For each of the 5(6) constellations, find:
(a) The average energy per symbol $E_S$ and the average energy per bit $E_b$.

(b) Assuming that the probability of error is dominated by errors between the nearest points in the constellation, find $d_{\text{min}}$, the smallest distance between two points. Then find the probability that a given symbol is received with error, using the approximation

$$P_e(\text{symbol}) \approx \frac{1}{2} \text{erfc} \left( \frac{d_{\text{min}}}{2 \sqrt{N_0}} \right)$$


2. If we want to achieve a particular $P_e(\text{symbol})$, sort the constellations from most energy-efficient to least. Justify your answer.

3. Now assume that the symbols are grey–coded, meaning that (practically) every symbol error results in one bit error only. For each of the 5(6) constellations:

(a) Find the (approximate) probability $P_e(\text{bit})$ that a given received bit is in error, as a function of $E_b/N_0$.

(b) Assuming $E_b/N_0 = 13$ dB, find the numeric value of $P_e(\text{bit})$ (hint: use the MATLAB function $\text{erfc}$).

4. When $E_b/N_0 = 13$ dB, sort the constellations from lowest $P_e(\text{bit})$ to highest.

**Solution**

1. The parameters are:

<table>
<thead>
<tr>
<th>System</th>
<th>BPSK</th>
<th>QPSK</th>
<th>8QAM</th>
<th>8QAM-Mod</th>
<th>8PSK</th>
<th>16QAM</th>
</tr>
</thead>
<tbody>
<tr>
<td>bits per symbol</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>$E_S$</td>
<td>1</td>
<td>1</td>
<td>4</td>
<td>21</td>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td>$E_b$</td>
<td>1</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{7}{4}$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{5}{2}$</td>
</tr>
<tr>
<td>$d_{\text{min}}$</td>
<td>2</td>
<td>$\sqrt{2}$</td>
<td>1</td>
<td>2</td>
<td>$2 \sin \frac{\pi}{8}$</td>
<td>2</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>1</td>
<td>1</td>
<td>$\frac{1}{\sqrt{2}}$</td>
<td>$\frac{3}{\sqrt{2}}$</td>
<td>$\sqrt{3} \sin \frac{\pi}{8}$</td>
<td>$\sqrt{\frac{3}{2}}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\approx 0.7071$</td>
<td>$\approx 0.7559$</td>
<td>$\approx 0.6628$</td>
<td>$\approx 0.6325$</td>
</tr>
</tbody>
</table>

We find $\alpha$ like so:

$$\frac{1}{2} \text{erfc} \left( \frac{d_{\text{min}}}{2 \sqrt{N_0}} \right) = \frac{1}{2} \text{erfc} \left( \alpha \sqrt{\frac{E_b}{N_0}} \right) \Rightarrow \alpha = \frac{d_{\text{min}}}{2 \sqrt{E_b}}$$
2. The higher the $\alpha$, the lower the $E_b/N_0$ needed to achieve a particular $P_e$(symbol). Therefore, constellations with higher $\alpha$ are more energy efficient for a given $P_e$(symbol):
BPSK = QPSK > 8QAM–Mod > 8QAM > 8PSK > 16QAM.

3. For grey-coded constellations:
We have $P_e$(bit) = $P_e$(symbol)/(bits per symbol):

<table>
<thead>
<tr>
<th>System</th>
<th>$P_e$(bit)</th>
<th>$P_e$(bit) when $E_b/N_0 = 13$ dB $\approx 19.95$</th>
</tr>
</thead>
<tbody>
<tr>
<td>BPSK</td>
<td>$\frac{1}{2}$ erfc $\left( \sqrt{\frac{E_b}{N_0}} \right)$</td>
<td>1.333 $\cdot 10^{-10}$</td>
</tr>
<tr>
<td>QPSK</td>
<td>$\frac{1}{4}$ erfc $\left( \sqrt{\frac{E_b}{N_0}} \right)$</td>
<td>6.665 $\cdot 10^{-11}$</td>
</tr>
<tr>
<td>8QAM</td>
<td>$\frac{1}{6}$ erfc $\left( \frac{1}{\sqrt{2}} \sqrt{\frac{E_b}{N_0}} \right)$</td>
<td>1.323 $\cdot 10^{-6}$</td>
</tr>
<tr>
<td>8QAM–Mod</td>
<td>$\frac{1}{6}$ erfc $\left( \sqrt{\frac{1}{2}} \sqrt{\frac{E_b}{N_0}} \right)$</td>
<td>2.991 $\cdot 10^{-7}$</td>
</tr>
<tr>
<td>8PSK</td>
<td>$\frac{1}{6}$ erfc $\left( \sqrt{\frac{1}{2}} \sqrt{\frac{E_b}{N_0}} \right)$</td>
<td>4.709 $\cdot 10^{-6}$</td>
</tr>
<tr>
<td>16QAM</td>
<td>$\frac{1}{8}$ erfc $\left( \sqrt{\frac{2}{5}} \sqrt{\frac{E_b}{N_0}} \right)$</td>
<td>8.078 $\cdot 10^{-6}$</td>
</tr>
</tbody>
</table>

(Note: do not forget to convert $E_b/N_0$ from dB to linear scale.)

4. QPSK > BPSK > 8QAM–Mod > 8QAM > 8PSK > 16QAM.