# Laboratory #1

## Time and Frequency Response of a Butterworth Filter

SYSC 3503 Communication Theory II

Department of Systems and Computer Engineering
Faculty of Engineering
Carleton University

# 1 Purpose

The purpose of this laboratory is to explore the input-output relationship of a linear time invariant (LTI) system; in particular, an 8th order Butterworth filter will be examined in this laboratory.

# 2 Objective

The objective of this laboratory is to reinforce the understanding of the frequency domain representation of signals and systems, through the use of the Fourier series and power spectral density.

# 3 Prelaboratory Exercises

## 3.1 Sinusoidal input

P1. Find the Fourier series and the power spectral density (PSD) for a $f_0$ Hz sinusoidal signal, $x(t) = \cos 2\pi f_0 t$, and sketch both.

P2. Plot the magnitude response of an 8th-order Butterworth filter from 0 Hz to 3 Hz with a cutoff frequency $f_{\text{cut}} = 1$ Hz. Use some software to do this (e.g. MATLAB). The magnitude response of an $N$th-order Butterworth filter is:

$$|H(f)| = \frac{1}{\sqrt{1 + (f/f_{\text{cut}})^{2N}}}$$

P3. Sketch the PSD of the output signal, $y(t)$, for input cosine signals of $f_0 = 0.5$ Hz, 1 Hz, and 3 Hz.

$$x(t) \longrightarrow \boxed{H(f)} \longrightarrow y(t)$$
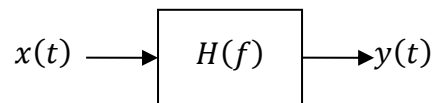
P4. How does the ratio of the amplitudes of $y(t)$ and $x(t)$ relate to $|H(f)|$ for different sinusoid frequencies?

## 3.2 Square wave input

P5. Find an expression for the Fourier series expansion of the 1 Hz square wave $x(t)$ shown below. Find the values of the Fourier series coefficients, $c_k$, for $k = 0,1,\dots,10$.
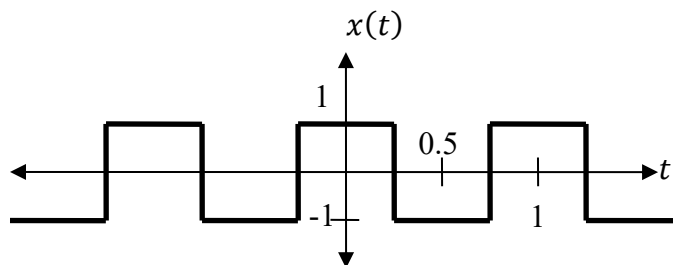


P6. Find an expression for the PSD of the square wave (in terms of $c_k$), and sketch for frequencies below 10 Hz in dB.

P7.   Sketch the PSD in dB (up to 10 Hz) for the output, $y(t)$, of an 8th-order Butterworth filter with (a) $f_{cut} = 2$ Hz and (b) $f_{cut} = 4$ Hz.

# 4   Laboratory

## 4.1   Sinusoidal input

1. In MATLAB, generate a sampled version of the analogue sinusoidal signal
$$x(t) = A\cos(2\pi f_0 t)$$
with an amplitude of $A = 1$ and a frequency of $f_0 = 0.5$ Hz. Use a sampling frequency of $f_s = 50$ Hz, and sample the signal for 10 seconds. **Plot $x(t)$ vs. $t$ and include the plot in your lab report**. To save a plot, select "Edit|Copy Figure" from the menu bar of the figure you want to copy, then select the "Paste" option in an open Word document.

2. Generate the difference equation coefficients of an 8th-order Butterworth lowpass filter with a cutoff frequency of $f_{cut} = 1$ Hz by using the MATLAB command

```
>> [B, A] = butter(N, 2*fcut/fs);
```

where `fcut` is the cutoff frequency (use 1 Hz), `fs` is the sampling frequency (use 50 Hz), and `N` is the filter order (use `N` = 8). The results `B` and `A`, are the difference equation coefficients. The filter output y[n] is related to the input x[n] by the difference equation
$$y[n] = \left(\begin{matrix} B(1)x[n] + B(2)x[n-1] + \cdots + B(N+1)x[n-N] \\ -A(2)y[n-1] - \cdots - A(N+1)y[n-N] \end{matrix}\right) \Big/ A(1) \ .$$

3. Apply the Butterworth filter to the signal you generated in Step 1 by using the MATLAB command

```
>> y = filter(B, A, x);
```

**Plot the output signal $y(t)$ vs. $t$ and include the plot in your lab report.** It should look similar to the graph below.
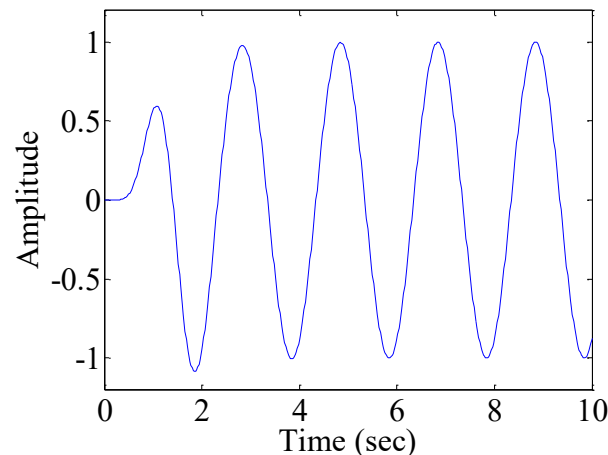


Observe that the output signal becomes what is essentially a sine wave, after an initial transition interval while the filter output adapts to the abrupt change in the input (it is assumed that the input is 0 for all $t < 0$, and then the input sine wave starts at $t = 0$).

4. Repeat Steps 1 to 3, first with a sinusoid of frequency $f_0 = 1$ Hz, and then with $f_0 = 3$ Hz. **Include plots of the input and the output signals in your lab report.**

5. Record the amplitudes of the steady state output sinusoid after the transition interval for the following input frequencies:

| Frequency (Hz) | 0.50 | 0.75 | 1.00 | 1.25 | 1.50 | 1.75 | 2.00 | 2.25 | 2.50 | 3.00 |
|---|---|---|---|---|---|---|---|---|---|---|
| Output Amplitude | | | | | | | | | | |

In prelab exercise P4 you came up with a relationship between the ratio of amplitudes of the input and output sinusoids, and $|H(f)|$. In the table above you recorded a number of output amplitudes for a number of frequencies, using an input sinusoid with constant amplitude. Determine the values of $|H(f)|$ for these frequencies, plot them, and compare this plot to your plot from prelab exercise P2. **Include the above table and your plot in your lab report, along with a relevant discussion of the results.**

## 4.2 Square wave input

1. Use the MATLAB `square` function (type `help square` for instructions) to generate a square wave, $x(t)$, with a period of 1 second and duration of 10 seconds. Use a sampling frequency of 1000 Hz. **Plot $x(t)$ vs. $t$ and include the plot in your lab report.** It should look like the signal shown below:
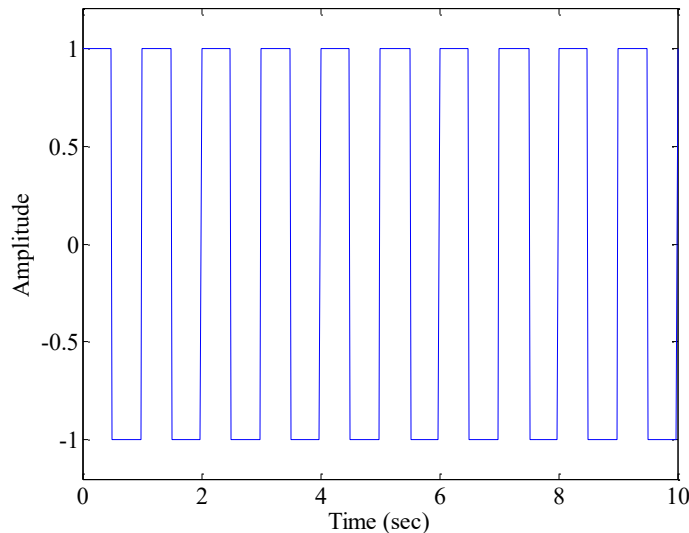
2. Estimate the PSD of $x(t)$ by using the discrete Fourier transform, as follows:

```
% Determine the length of x[n]
Ns = length(x);

% Create values for k
k = -floor(Ns/2):floor((Ns-1)/2);

% Create values for f
f = k / (Ns*Ts);

% Get the DFT of x[n] and approximate the CTFT from the DFT
X = Ts * fftshift(fft(x));

% Plot the PSD of X(f) in dB
plot(f, 20*log10(abs(X)));
```

In the above, `x` is the discrete-time representation of $x(t)$, `X` is the discrete frequency-domain representation of $X(f)$ and `Ts` is the sampling period ($T_s = 1/f_s$).

You should use the MATLAB `axis` command to limit the graph to frequencies between 0 and 10 Hz, and to amplitudes in the range from -40 dB to +20 dB. **Include this plot in your lab report.**

3. Compare the theoretical PSD of the square wave with the observed PSD. Because the PSD is estimated based on a truncated square wave (the duration is only 10 seconds) instead of an infinite-duration periodic signal, the impulses in the theoretical PSD (which have infinite amplitude but finite area) appear as finite-amplitude spikes in the estimated PSD. The amplitudes of these spikes are proportional to the areas of the impulses in the theoretical results, but there will be a constant shift in the dB values (in non-dB values, this corresponds to a constant multiplicative factor). Estimate the shift by finding the difference between the theoretical value of the first peak and the observed first peak. **Provide the value of this shift in your lab report. Remove the constant shift from the observed values and provide a table comparing these values with the values of $|c_k|^2$ in dB from prelab exercise P6.**

4. Apply an 8$^{\text{th}}$-order Butterworth filter with a cutoff frequency of $f_{\text{cut}} = 2$ Hz to the signal you generated in Step 1. **Plot the output signal $y(t)$ vs. $t$ and include the plot in your lab report. Plot the PSD of $y(t)$ from 0 to 10 Hz, and include the plot in your report. Compare the theoretical output PSD with the observed PSD.**

5. Repeat Step 4 with a cutoff frequency of $f_{\text{cut}} = 4$ Hz.

6. **Explain what you see in the observed output waveforms $y(t)$ for the two cutoff frequencies.**