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Stochastic Processes 94553

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1 (a) Random variables X_i, $i = 1, \ldots, n$ are Gauassian and independent, and thus jointly Gaussian. Any linear combination of them is therefore Gaussian, which means each Y_i, $i = 1, \ldots, n$, is Gaussian, and fully described by its mean and variance. Now it can be seen that $m_{Y_i} = im$ and $\sigma^2_{Y_i} = i\sigma^2$.

(b) It is easy to see that $(Y_1, \ldots, Y_n)$ is a linear transformation of $(X_1, \ldots, X_n)$. Since X_i's are jointly Gaussian, Y_i's will also be jointly Gaussian. In particular, the joint distribution of Y_i and Y_{i+1} is fully described by the means, variances and the covariance. The means and variances were given in part (a). The covariance is equal to:

$$\sigma^2_{Y_i,Y_{i+1}} = E(\tilde{Y_i}\tilde{Y_{i+1}}) = E(\tilde{Y_i}^2) + E(\tilde{Y_i}\tilde{X_{i+1}}) = i\sigma^2 + E(\tilde{Y_i})E(\tilde{X_{i+1}}) = i\sigma^2 ,$$

where in the second last step, we have used the independence of Y_i and X_{i+1}.

(c) Let $Z = \sum_{j=1}^{i-1} X_j$. Then Z is Gaussian and independent of X_i. They are thus jointly Gaussian. Now, $Y_i = Z + X_i$, which means (Y_i, X_i) is a linear transformation of (Z, X_i), and therefore X_i and Y_i are jointly Gaussian. It follows that $X_i|Y_i = m$ is Gaussian. From the attached formulas, we know that the expected value and variance of this random variable are equal to

$$E(X_i|Y_i = m) = \rho\frac{\sigma_{X_i}}{\sigma_{Y_i}}(m - m_{Y_i}) + m_{X_i} = \frac{\rho m}{\sqrt{i}}(1 - i) + m ,$$

and

$$\sigma^2_{X_i|Y_i=m} = \sigma^2_{X_i}(1 - \rho^2) = \sigma^2(1 - \rho^2) ,$$

respectively, where

$$\rho = \frac{\sigma_{X_i,Y_i}}{\sigma_{X_i}\sigma_{Y_i}} = \frac{E(\tilde{Y_i}\tilde{X_i})}{\sigma\sqrt{i}\sigma} = \frac{E(\tilde{X_i}^2)}{\sqrt{i}\sigma^2} = \frac{\sigma^2}{\sqrt{i}\sigma^2} = \frac{1}{\sqrt{i}} .$$

So, $W = (X_i|Y_i = m)$ has the distribution $N(\frac{m}{i}, \sigma^2(1 - \frac{1}{i}))$. It is now easy to see that

$$P(W > 0) = 1 - Q(\frac{m}{\sigma\sqrt{i(i-1)}}) .$$

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2 (a) Starting by the integration of the joint pdf of X and Y (which is the product of the marginal pdfs) over the region $X - Y \leq z$, one can obtain the cdf $F_Z(z)$. Taking the derivative of $F_Z(z)$, we then obtain:

$$f_Z(z) = \begin{cases} (1 - e^{-\lambda})e^{\lambda z} & z \leq 0; \\ 1 - e^{\lambda(z-1)} & 0 \leq z \leq 1; \\ 0 & z > 1 . \end{cases}$$

(b) $E(Z^2|Y = 1) = E[(X - Y)^2|Y = 1) = E[(X - 1)^2|Y = 1] = E[(X - 1)^2] = \frac{1}{3}$, where in the second last step, we have used the independence of X and Y.

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3 (a) Refer to the textbook or your lecture notes.

(b) It is easy to see that $M_n = \frac{1}{n}\sum_{i=1}^n X_i$ has a Gaussian distribution with mean μ and variance σ^2/n. We thus have

$$P[M_n - z\frac{\sigma}{\sqrt{n}} \leq \mu \leq M_n + z\frac{\sigma}{\sqrt{n}}] = P[-z \leq \frac{M_n - \mu}{\sigma/\sqrt{n}} \leq z] = 1 - 2Q(z) = \beta\,.$$

Therefore the $\beta \times 100\%$ confidence interval for μ is $[M_n - z\frac{\sigma}{\sqrt{n}}, M_n + z\frac{\sigma}{\sqrt{n}}]$, where $z = Q^{-1}(\frac{1-\beta}{2})$.