Carleton University
Department of Systems and
Computer Engineering

Midterm Exam

Stochastic Processes
94.553, February 26, 2002
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Time allowed: 1.5 hours.
Closed book exam. Some formulas which may be helpful are attached.
[35] constitutes full marks.
There is no correlation between the order and the difficulty of the problems.
Attempt all the three questions. Justify all your answers.

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1. A discrete random variable X can take three values 0, 1 and 2, with probabilities satisfying $P(X=0) = 2P(X=1) = 4P(X=2)$. For another random variable Y, the conditional probability density functions (pdf) are defined by

$$f_Y(y|X=0) = f_Y(y|X=1) = f_Y(y|X=2) = \begin{cases} 1 & 0 \le y \le 1; \\ 0 & \text{otherwise.} \end{cases}$$

3 marks (a) Find the probability mass function (pmf) of X and the pdf of Y.

3 marks (b) Find the joint pdf $f_{X,Y}(x,y)$.

2 marks (c) Are X and Y independent? why?

3 marks (d) Compute the expected value $E(XY)$ and the variance of X.

2 marks (e) Find the probability $P(X > 0.5|Y = 0.5)$.

3 marks 2. a) Random variables X and Y are binary and independent with $P(X=1) = p$ and $P(Y=1) = q$. Let $Z = X + Y \bmod 2$ (binary exclusive OR). Find the distribution of Z in terms of p and q.

4 marks b) Random variables X and Y are jointly Gaussian with means $E(X) = 0, E(Y) = 1$, and the following covariance matrix

$$K = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}.$$

Find the distribution of $Z = X + 2Y$.

6 marks c) Explain the law of large numbers (LLN) and the difference between the strong and the weak LLN. Also, explain how LLN can be used to estimate the probability $P(A)$ of an event A.

3. In a digital communication system, a bit B is transmitted over the channel by being mapped to the signal $X = (-1)^B$. At the output of the channel, the received value is $Y = X + N$, where N is a Gaussian random variable with zero mean and variance σ^2, independent of X. The received value Y is then quantized to one of the three possible values +1, 0 and -1, according to the following function

$$Z(Y) = \begin{cases} -1 & Y < -\epsilon; \\ 0 & -\epsilon \le Y \le \epsilon; \\ +1 & Y > \epsilon\,. \end{cases}$$

where $0 < \epsilon < 1$ is a constant. Assuming that B is equally likely to be 0 or 1, 4 marks (a) Find the distribution of Z at the output of the quantizer in terms of Q function.

(b) To transmit the information reliably over this channel, a repetition code of length 5 is used, i.e., bit B is transmitted 5 times over the channel independently (noise random variables are independent). At the receiver, the output is decoded as 0 or 1 if out of the 5 quantized received values, the majority of nonzero values are +1 or -1, respectively, and there are at most 3 zeros among the 5 values. Otherwise, the output is decoded as "erasure," and the receiver requests the retransmission of the information. (note that based on this decoding rule, the output combinations (+1, +1, -1, -1, 0), or (+1, 0, 0, 0, 0) are declared as erasure.)

5 marks i. Find the probability that the receiver requests a retransmission.

Some Important Random Variables:

(a) Binomial

$$\mathcal{S}_X = \{0, 1, 2, \ldots, n\}$$

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}, k = 0, 1, \ldots, n$$

$$E[X] = np; \quad VAR[X] = np(1-p); \quad \Phi_X(\omega) = (1 - p + pe^{j\omega})^n$$

(b) Poisson

$$\mathcal{S}_X = \{0, 1, 2, 3, \ldots\}$$

$$p_X(k) = \frac{\alpha^k e^{-\alpha}}{k!}, k = 0, 1, 2, \ldots \text{ and } \alpha > 0$$

$$E[X] = \alpha; \quad VAR[X] = \alpha; \quad \Phi_X(\omega) = exp(\alpha(e^{j\omega} - 1))$$

(c) Exponential

$$\mathcal{S}_X = [0, \infty)$$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0; \\ 0 & \text{otherwise.} \end{cases} \quad \text{where } \lambda > 0$$

$$E[X] = \frac{1}{\lambda}; \quad VAR[X] = \frac{1}{\lambda^2}; \quad \Phi_X(\omega) = \frac{\lambda}{\lambda - j\omega}$$

(d) Gaussian (or Normal)

$$\mathcal{S}_X = (-\infty, \infty)$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-m)^2/(2\sigma^2)}$$

$$E[X] = m; \quad VAR[X] = \sigma^2; \quad \Phi_X(\omega) = e^{jm\omega - \sigma^2\omega^2/2}$$

Some definitions:

(a) $COV[X, Y] = E[XY] - E[X]E[Y]$.

(b) $Q(z) = \int_z^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$.

(c) The random variables $X_1, \ldots, X_n$ are called jointly Gaussian if their joint pdf is given by

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{\exp\{-\frac{1}{2}(\mathbf{x} - \mathbf{m})^T K^{-1} (\mathbf{x} - \mathbf{m})\}}{\sqrt{(2\pi)^n \det(K)}},$$

where $\mathbf{x}$ and $\mathbf{m} = (E[X_1], \ldots, E[X_n])^T$ are column vectors, and K is the covariance matrix.

(d) If X and Y are jointly Gaussian, the conditional pdf's $f_X(x|y)$ and $f_Y(y|x)$ are also Gaussian, e.g., $f_X(x|y)$ is Gaussian with mean $m_X + \rho(\sigma_X/\sigma_Y)(y - m_Y)$ and variance $\sigma_X^2(1 - \rho^2)$.