Carleton University
Department of Systems and Computer Engineering

FINAL EXAMINATION, APRIL 15, 2002
Stochastic Processes — 94.553
Examiner: Amir H. Banihashemi

(a) Check that you have all the three [3] pages.

(b) Time allowed: 3 hours.

(c) Open-book exam.

(d) [65] constitutes full marks. The value of each question is indicated beside the question.

(e) There is no correlation between the order and the difficulty of questions.

(f) You can use the results obtained during the lectures and in the textbook provided that you properly refer to them. Any other result needs to be derived.

(g) **Attempt all the three questions. Justify all your answers.**

(h) In this exam: pdf = probability density function
wss = wide sense stationary
psd = power spectral density
LSI = linear shift invariant

1. Short Problems:

[4] (a) Show that if R is the correlation matrix for the real random vector $\underline{X} = (X_1, \ldots, X_n)$ and R^{-1} is its inverse, then $E(\underline{X} R^{-1} \underline{X}^T) = n$.

[5] (b) Suppose that $X(t)$ is a wss Gaussian random process with $E[X(t)] = 2$ and auto-covariance function $C_X(\tau) = e^{-0.5|\tau|}$. Find the probability that $|X(2) - X(1)| \leq 1$ in terms of $Q(.)$ function.

[6] (c) Random process $X(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$ is wss. Prove that for random variables A and B, we must have $E(A) = E(B) = E(AB) = 0$ and $E(A^2) = E(B^2)$.

[4] (d) Let $X(t) = ae^{j(\Omega t + \Theta)}$, where Ω is a random variable with pdf $f_\Omega(\omega)$, Θ is a random variable uniformly distributed in $[0, 2\pi)$ and independent of Ω, and a is a complex constant. Show that $X(t)$ is wss by finding its mean and autocorrelation function (find the autocorrelation function in terms of the characteristic function $\Phi_\Omega(.)$).

[6] (e) A Gaussian wss random process $X(t)$ with zero mean and autocorrelation function $R_X(\tau)$ is applied to a square-law system whose output equals $Y(t) = X^2(t)$. i) Find the mean $E[Y(t)]$ and the pdf $f_{Y(t)}(y)$ of the output $Y(t)$. ii) Find the autocorrelation function $R_Y(t, t-\tau)$ of the output as a function of $R_X(.)$ (Hint: For jointly Gaussian random variables V and W, we have $E(V^2W^2) = E(V^2)E(W^2) + 2E^2(VW)$).

[6] (f) A Poisson process $N(t)$ with λ events per time unit is passed through a differentiator. If the output is denoted by $Z(t)$, and assuming that the system has reached a steady-state, find $E[Z(t)]$, $R_{ZN}(t_1, t_2)$ and $R_Z(t_1, t_2)$.

[3] (g) Show that for a real random process X_n, if $R_X(k_1, k_2) = b_{k_1}\delta_{k_1-k_2}$ and $s = \sum_{n=0}^{N} a_n X_n$, then $E(s^2) = \sum_{n=0}^{N} a_n^2 b_n$.

[4] (h) Show that the process $X(t) = Ae^{j(\omega t + \Theta)}$, in which Θ is uniformly distributed in $[0, 2\pi)$ and is independent of A, is not <u>correlation</u> ergodic.

[5] 2. (a) Show that for an LSI system with input X_n, output Y_n, and unit-sample response h_n, if $R_X(k_1, k_2) = b_{k_1}\delta_{k_1-k_2}$, then

$$E[|Y_n|^2] = b_n * |h_n|^2 = \sum_{k=-\infty}^{\infty} b_{n-k}|h_k|^2 \ .$$

[4] (b) Using the result of part (a), find $E[Y_n^2]$ if

$$8Y_n - 6Y_{n-1} + Y_{n-2} = X_n \ ,$$

where the input X_n has an autocorrelation function $R_X(k) = 4\delta_k$, and is applied to the system at $n = 0$ (assume that the system is in zero initial condition at $n = 0$).

3. Suppose that $V(t)$ is a wss random process. Define $W(t) = V(t)e^{j\Theta}$, where Θ is a random variable uniformly distributed in $[-\pi, \pi)$, and independent of $V(t)$.

[3] (a) Prove that the random process $W(t)$ is orthogonal to its complex conjugate $W^*(t)$.

(b) Assuming that $V(t)$ has the psd $S_V(f)$ given in Fig. 1,

[4] i. Find $R_V(\tau)$ and $R_W(t, t-\tau)$.

[3] ii. Find and plot the psd of $X(t) \triangleq Re[W(t)]$ and that of $Y(t) \triangleq Im[W(t)]$.

[4] iii. Find the cross-correlation function $R_{XY}(t, t-\tau)$.

[4] iv. Find and plot the psd of $Z(t) = X(t)\cos(4\pi t) - Y(t)\sin(4\pi t)$.

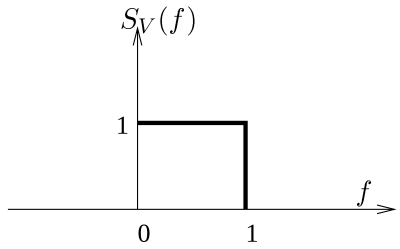

Figure 1: Psd of $V(t)$ in problem 3