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Stochastic Processes, 94.553

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Problem Set #4 Solutions

- **Textbook:** Ch. 4: 108; Ch. 5: 3, 5, 6, 9, 11, 19, 21, 29, 37.

4.108 (a) We have

$$K_X = \begin{pmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{pmatrix},$$

with orthonormal eigenvectors $\frac{1}{\sqrt{2}}(1,1)^T$ and $\frac{1}{\sqrt{2}}(-1,1)^T$. Therefore, a choice of linear transformation is

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

(b) The following matrix transforms two consecutive blocks of $\underline{\mathbf{X}} = (\mathbf{X}_i, \mathbf{X}_{i+1})^T$ to the corresponding blocks of $\underline{\mathbf{Y}}$:

$$A' = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

We thus have $K_{\underline{Y}} = A' K_{\underline{X}} A'^T$, and it is easy to see from $K_{\underline{Y}}$ that the components of $\mathbf{Y}_i$ and $\mathbf{Y}_{i+1}$ are not independent, e.g., $\sigma_{Y_1,Y_3} = \frac{\rho\sigma^2}{2}$.

5.3 $E(\sum X_i) = \sum E(X_i) = n\mu$.

We can display the variances and covariances in a covariance matrix, where the i,jth entry is $COV(X_i, X_j)$:

$$K = \begin{bmatrix} \sigma^2 & \rho\sigma^2 & \rho^2\sigma^2 & \cdots & \rho^{n-1}\sigma^2 \\ \rho\sigma^2 & \sigma^2 & \rho\sigma^2 & \cdots & \rho^{n-2}\sigma^2 \\ \rho^2\sigma^2 & \rho\sigma^2 & \sigma^2 & \cdots & \rho^{n-3}\sigma^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho^{n-1}\sigma^2 & \rho^{n-2}\sigma^2 & \rho^{n-3}\sigma^2 & \cdots & \sigma^2 \end{bmatrix}$$

Then $VAR(\sum X_i)$ is the sum of the entries in K:

$$\begin{aligned} VAR(S_n)/\sigma^2 &= 2\sum_{i=0}^{n-1}(n-i)\rho^i - n \\ &= \frac{2n(1-\rho^n)}{1-\rho} - n - 2\rho\sum_{i=0}^{n-1} i\rho^{i-1} \\ &= \frac{2n(1-\rho^n)}{1-\rho} - n - 2\rho\frac{d}{d\rho}\sum_{i=0}^{n-1}\rho^i \\ &= \frac{2n(1-\rho^n)}{1-\rho} - n - 2\rho\frac{d}{d\rho}\frac{1-\rho^n}{1-\rho} \end{aligned}$$

$$
\begin{aligned}
&= \frac{2n(1-\rho^n)}{1-\rho} - n - 2\rho\frac{(-n\rho^{n-1}(1-\rho)-(1-\rho^n)(-1)}{(1-\rho)^2} \\
&= \frac{\rho^{n+1}-n\rho^2-2\rho+n}{(1-\rho)^2} .
\end{aligned}
$$

Thus

$$
VAR(S_n) = \sigma^2\frac{\rho^{n+1}-n\rho^2-2\rho+n}{(1-\rho)^2}.
$$

5.5

$$
\Phi_{S_k}(\omega) = (\frac{1}{1-2j\omega})^{n_1/2}(\frac{1}{1-2j\omega})^{n_2/2}\cdots(\frac{1}{1-2j\omega})^{n_k/2} = (\frac{1}{1-2j\omega})^{(n_1+\cdots+n_k)/2} ,
$$

and thus S_k is a chi-square r.v. with $n = n_1 + \cdots + n_k$.

5.6 (a) From Ex. 3.26, X_i^2 is chi-square with one degree of freedom. From Prob. 5.5, S_n is then chi-square with n degrees of freedom.

(b)

$$
T_n = \sqrt{S_n} \Longrightarrow f_{T_n}(x) = \frac{f_{S_n}(x^2)}{\frac{1}{2}|(x^2)^{-1/2}|} = 2xf_{S_n}(x^2) .
$$

Now using the fact that S_n is chi-square, we have

$$
f_{T_n}(x) = \frac{2x(x^2)^{\frac{n-2}{2}}e^{-x^2/2}}{2^{n/2}\Gamma(n/2)} = \frac{x^{n-1}e^{-x^2/2}}{2^{n/2-1}\Gamma(n/2)} , \quad x > 0 .
$$

(c) $f_{T_2}(x) = xe^{-x^2/2}$, $x > 0$, which is a Rayleigh distribution.

(d) $f_{T_3}(x) = \frac{x^2e^{-x^2/2}}{2^{1/2}\Gamma(3/2)} = \sqrt{\frac{2}{\pi}}x^2e^{-x^2/2} , \quad x > 0 .$

5.9

$$
\begin{aligned}
\Phi_{M_n}(\omega) &= E[\exp(j\omega M_n)] \\
&= E[\exp(j\omega\frac{1}{n}\sum_{i=1}^{n} X_i)] \\
&= \prod_{i=1}^{n} E[\exp(j\omega X_i/n] \\
&= (\Phi_X(\omega/n))^n
\end{aligned}
$$

5.11 $G_{S_k}(z) = G_{X_1}(z)\cdots G_{X_k}(z) = e^{\alpha_1(z-1)}\cdots e^{\alpha_k(z-1)} = e^{(\alpha_1+\cdots+\alpha_k)(z-1)}$. Thus, S_k is Poisson with the mean $\alpha_1 + \cdots + \alpha_k$.

5.19 Using the Chebyshev inequality, we have

$$
P(|\frac{S_n}{n} - \mu| > \epsilon) \le \frac{\mathrm{VAR}(S_n)}{n^2\epsilon^2} .
$$

Using the covariance function of Problem 5.2, it can be seen that $\text{VAR}(S_n) = n\sigma^2 + 2(n-1)\rho\sigma^2$, and thus the RHS of the above inequality tends to zero as $n \longrightarrow \infty$, which means that the WLLN holds.

5.21 (a) Since $M_n = (1/n)\sum_{j=1}^n X_j$, we have

$$\begin{aligned}
RHS &= n(M_n - \mu)^2 + \sum_{j=1}^n (X_j - M_n)^2 \\
&= nM_n^2 - 2n\mu M_n + n\mu^2 + \sum_{j=1}^n (X_j^2 - 2M_n X_j + M_n^2) \\
&= nM_n^2 - 2n\mu M_n + n\mu^2 + (\sum_{j=1}^n X_j^2) - 2nM_n^2 + nM_n^2 \\
&= (\sum_{j=1}^n X_j^2) - 2n\mu M_n + n\mu^2 \\
&= \sum_{j=1}^n (X_j^2 - 2\mu X_j + \mu^2) \\
&= \sum_{j=1}^n (X_j - \mu)^2 \\
&= LHS
\end{aligned}$$

(b) Using the result of part (a), and observing that $VAR(M_n) = E[(M_n - \mu)^2] = \sigma^2/n$,

$$\begin{aligned}
E\left[k\sum_{j=1}^n (X_j - M_n)^2\right] &= kE\left[\sum_{j=1}^n (X_j - \mu)^2 - n(M_n - \mu)^2\right] \\
&= k\sum_{j=1}^n E[(X_j - \mu)^2] - nkE[(M_n - \mu)^2] \\
&= nk\sigma^2 - nk\sigma^2/n \\
&= k(n-1)\sigma^2
\end{aligned}$$

(c) Using the result of part (b) with $k = 1/(n-1)$ yields $E[V_n^2] = \sigma^2$.

(d) Using the result of part (b) with $k = 1/n$ yields

$$E[(n-1)V_n^2/n] = (n-1)\sigma^2/n = \sigma^2 - \underbrace{\sigma^2/n}_{\text{bias}}.$$

5.29 The total number of errors in 100 bit transmissions, S_{100}, has a binomial distribution with mean $m = 100p = 15$ and variance $\sigma^2 = 100p(1-p) = 12.75$. Using the CLT, S_{100} can be approximated by a normal distribution with the same mean and variance. We therefore have

$$P(S_{100} \leq 20) = 1 - P(\frac{S_{100} - 15}{\sqrt{12.75}} > \frac{20 - 15}{\sqrt{12.75}}) \approx 1 - Q(1.4) = .92\,.$$

5.37 Let $Y_1, \ldots, Y_{10}$ denote the sample means of the 10 batches. The sample mean and the sample variance of Y_is are thus equal to:

$$M_{10} = \frac{\sum Y_i}{10} = .4980, \quad V_{10}^2 = \frac{1}{9}\sum (Y_i - M_{10})^2 = .0014\ .$$

Now for a confidence level of 95%, and $n-1=9$, we have $z_{\alpha/2,n-1} = 2.262$. Thus the 95% confidence interval for $E(Y_i) = p$ is $(M_{10} - 2.262\sqrt{\frac{.0014}{10}}, M_{10} + 2.262\sqrt{\frac{.0014}{10}}) = (0.4712, 0.5248)$.