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Stochastic Processes, 94.553

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Problem Set #3 Solutions

- **Textbook:** Ch. 4: 4, 10, 14, 22, 25, 32, 46, 48, 51, 61, 78, 81.

4.4 (a)

$$p_{X_1,X_2}(i,j) = \begin{cases} 1/36 & 1 \le i \le 6, \quad 1 \le j \le 6 \\ 0 & \text{otherwise} \end{cases}$$

(b)

$$p_{X,Y}(i,j) = \begin{cases} 1/36 & i = j, \quad 1 \le i,j \le 6 \\ 2/36 & 1 \le i < j \le 6 \\ 0 & \text{otherwise} \end{cases}$$

	$j = 1$	2	3	4	5	6
$i = 1$	1/36	2/36	2/36	2/36	2/36	2/36
2	0	1/36	2/36	2/36	2/36	2/36
3	0	0	1/36	2/36	2/36	2/36
4	0	0	0	1/36	2/36	2/36
5	0	0	0	0	1/36	2/36
6	0	0	0	0	0	1/36

(c)

$$\begin{aligned} P_X(1) &= 1/36 + 5 * 2/36 = 11/36 \\ P_X(2) &= 1/36 + 4 * 2/36 = 9/36 \\ P_X(3) &= 1/36 + 3 * 2/36 = 7/36 \\ P_X(4) &= 1/36 + 2 * 2/36 = 5/36 \\ P_X(5) &= 1/36 + 1 * 2/36 = 3/36 \\ P_X(6) &= 1/36 \end{aligned}$$

In general, $P_X(k) = (13-2k)/36$, $1 \le k \le 6$. By symmetry, $P_Y(k) = P_X(7-k) = (2k-1)/36$.

4.10 (a) $k = 1$, since

$$\begin{aligned} 1/k &= \int_0^1 \int_0^1 (x+y)\,dy\,dx \\ &= \int_0^1 (xy + y^2/2)\Big|_0^1 dx \\ &= \int_0^1 (x + 1/2)\,dx \\ &= 1/2 + 1/2 = 1. \end{aligned}$$

(b) The cdf is zero outside the first quadrant. In the first quadrant, after integrating the pdf, we obtain

$$F_{XY}(x,y) = \begin{cases} xy(x+y)/2 & 0 \le x \le 1,\ 0 \le y \le 1 \\ y(y+1)/2 & x > 1,\ 0 \le y \le 1 \\ x(x+1)/2 & 0 \le x \le 1,\ y > 1 \\ 1 & x > 1,\ y > 1 \end{cases}$$

(c) $F_X(x) = 0, x < 0$. $F_X(x) = \lim_{y\to\infty} F_{X,Y}(x,y) = x(x+1)/2$ for $0 \le x \le 1$, and $F_X(x) = 1$ for $x > 1$. Thus, $f_X(x) = x + 1/2$, $0 \le x \le 1$, and $f_X(x) = 0$, otherwise. By symmetry, $f_Y(x) = f_X(x)$.

4.14 With $\rho = 0$, $f_{XY}(x,y) = \frac{1}{2\pi}e^{-(x^2+y^2)/2}$. Then,

$$\begin{aligned} P[X^2+Y^2 < R^2] &= \underbrace{\int\int}_{x^2+y^2<R^2} \frac{1}{2\pi}e^{-(x^2+y^2)/2}dydx \\ &= \int_{\theta=0}^{2\pi}\int_{r=0}^{R} \frac{1}{2\pi}e^{-r^2/2}rdrd\theta \\ &= -e^{-r^2/2}\Big|_0^R \\ &= 1-e^{-R^2/2} \end{aligned}$$

4.22 For X and Y to be independent, it is necessary (but not sufficient) that the joint pdf for X and Y be nonzero over a product-form region. For the regions of Fig. P4.1, this is never the case, hence X and Y are not independent for these regions.

4.25

(a) If $\rho = 0$ in Problem 15, we have

$$f_{X,Y}(x,y) = \frac{1}{\sqrt{2\pi}\sigma_1}e^{-\frac{(x-m_1)^2}{2\sigma_1^2}}\frac{1}{\sqrt{2\pi}\sigma_2}e^{-\frac{(x-m_2)^2}{2\sigma_2^2}} = f_x(x)f_Y(y)\ ,\ \forall x,y\ .$$

Thus X and Y are independent.

(b) $P[XY > 0] = \int_0^\infty\int_0^\infty f_{X,Y}(x,y)dxdy + \int_{-\infty}^0\int_{-\infty}^0 f_{X,Y}(x,y)dxdy$. Using the result of part (a), it is not then difficult to see that

$$P[XY > 0] = [1-Q(\frac{m_1}{\sigma_1})][1-Q(\frac{m_2}{\sigma_2})] + Q(\frac{m_1}{\sigma_1})Q(\frac{m_2}{\sigma_2})\ .$$

4.32 (a) From problem 4.11, we know that $f_{X,Y}(x,y) = 1/\pi$, $x^2+y^2 \le 1$, and zero otherwise. We also know that $f_X(x) = 2\sqrt{1-x^2}/\pi$, $-1 \le x \le 1$. Thus

$$\begin{aligned} f_{Y|X}(y|x) &= \frac{f_{X,Y}(x,y)}{f_X(x)} \\ &= \frac{1/\pi}{2\sqrt{1-x^2}/\pi} \\ &= \frac{1}{2\sqrt{1-x^2}},\quad -\sqrt{1-x^2} \le y < \sqrt{1-x^2}, \end{aligned}$$

i.e., the conditional pdf of Y given $X = x$ is uniform.

(b) From problem 4.11, we know that $f_{X,Y}(x,y) = 1/2$ inside the given region and zero otherwise. We also know that $f_X(x) = 1 - |x|$ for $-1 \le x \le 1$. Thus

$$\begin{aligned} f_{Y|X}(y|x) &= \frac{f_{X,Y}(x,y)}{f_X(x)} \\ &= \frac{1/2}{1-|x|} \\ &= \frac{1}{2(1-|x|)}, \quad |x| - 1 \le y \le 1 - |x|, \end{aligned}$$

i.e., the conditional pdf of Y given $X = x$ is uniform.

(c) From problem 4.11, we know that $f_{X,Y}(x,y) = 2$ inside the given region and zero otherwise. We also know that $f_X(x) = 2(1-x)$ for $0 \le x \le 1$. Thus

$$\begin{aligned} f_{Y|X}(y|x) &= \frac{f_{X,Y}(x,y)}{f_X(x)} \\ &= \frac{2}{2(1-x)} \\ &= \frac{1}{(1-x)}, \quad 0 \le y \le 1 - x, \end{aligned}$$

i.e., the conditional pdf of Y given $X = x$ is uniform.

Note: in any problem in which $f_{X,Y} = k$ (i.e., uniform over some region), the conditional pdf of Y given $X = x$ will be a function only of x; hence Y will have a uniform conditional distribution along the line $X = x$ where $f_{X,Y} \neq 0$. Once the limits are known, the conditional pdf is easily computed.

4.46 (a) $p(k_1, k_2) = \sum_{k_3=0}^{n-k_1-k_2} p(k_1,k_2,k_3) = \frac{n-k_1-k_2+1}{\binom{n+3}{3}}$, for $k_1, k_2 \ge 0, k_1 + k_2 \le n$.

(b) $p(k_1) = \sum_{k_2=0}^{n-k_1} \frac{n-k_1-k_2+1}{\binom{n+3}{3}}$. Let $j = n - k_1 - k_2 + 1$. Then

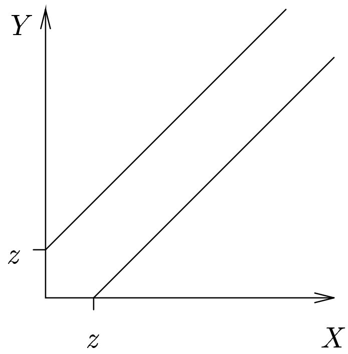
$$p(k_1) = \sum_{j=1}^{n-k_1+1} \frac{j}{\binom{n+3}{3}} = \frac{(n-k_1+2)(n-k_1+1)}{2\binom{n+3}{3}}, \text{ for } 0 \le k_1 \le n\,.$$

(c) $p(k_2, k_3|k_1) = \frac{p(k_1,k_2,k_3)}{p(k_1)} = \frac{2}{(n-k_1+2)(n-k_1+1)}$, for $k_2, k_3 \ge 0, k_2 + k_3 \le n - k_1$.

4.48 Since X is exponentially distributed, the marginal pdf for X is $f_X(x) = ae^{-ax}$, $x \ge 0$, $a > 0$. Although the problem does not state this, let us assume that Y has the same distribution as X. Then, since X and Y are independent, their joint pdf is given by

$$f_{X,Y}(x,y) = a^2 e^{-a(x+y)}, \ x \ge 0, \ y \ge 0.$$

Let $Z = |X - Y|$. For $z \le 0$, $P[Z \le z] = 0$. For $z > 0$, the region where $Z \le z$ is bounded by the lines indicated in the figure.



For $z > 0$,

$$\begin{aligned}
F_Z(z) &= \int_0^z \int_0^{x+z} f_{X,Y}(x,y)\,dy\,dx + \int_z^{\infty} \int_{x-z}^{x+z} f_{X,Y}(x,y)\,dy\,dx \\
&= \int_0^z ae^{-ax}(-e^{-ay})|_0^{x+z} dx + \int_z^{\infty} ae^{-ax}(-e^{-ay})|_{x-z}^{x+z} dx \\
&= \int_0^z (ae^{-ax} - ae^{-2ax-az})dx + \int_z^{\infty} (ae^{-2ax+az} - ae^{-2ax-az})dx \\
&= [-e^{-ax} + (1/2)e^{-az}e^{-2ax}]_0^z + [-(1/2)e^{az}e^{-2ax} + (1/2)e^{-az}e^{-2ax}]_z^{\infty} \\
&= -e^{-az} + (1/2)e^{-az}e^{-2az} + 1 - (1/2)e^{-az} + (1/2)e^{az}e^{-2az} - (1/2)e^{-az}e^{-2az} \\
&= 1 - e^{-az}\,.
\end{aligned}$$

Thus, $f_Z(z) = ae^{-az}$, $z \geq 0$, i.e., Z is also exponentially distributed with parameter a.

4.51 The cdf of $Z = X + Y$ can be computed, for $z \geq 0$, as

$$\begin{aligned}
F_Z(z) &= P[X + Y \leq z] \\
&= \int_{x=0}^{z/2} \int_{y=0}^{x} 2e^{-x}e^{-y} dy\,dx + \int_{x=z/2}^{z} \int_{y=0}^{z-x} 2e^{-x}e^{-y} dy\,dx \\
&= \int_{x=0}^{z/2} 2e^{-x}(1 - e^{-x})dx + \int_{x=z/2}^{z} 2e^{-x}(1 - e^{x-z})dx \\
&= \int_{x=0}^{z/2} (2e^{-x} - 2e^{-2x})dx + \int_{x=z/2}^{z} (2e^{-x} - 2e^{-z})dx \\
&= 2\int_{x=0}^{z} e^{-x}dx - 2\int_{x=0}^{z/2} e^{-2x}dx - 2e^{-z}\int_{x=z/2}^{z} dx \\
&= 2(1 - e^{-z}) - (1 - e^{-z}) - ze^{-z} \\
&= 1 - e^{-z} - ze^{-z}, \quad z \geq 0
\end{aligned}$$

Hence,

$$\begin{aligned}
f_Z(z) &= dF_Z(z)/dz \\
&= e^{-z} - e^{-z} + ze^{-z} \\
&= ze^{-z}, \quad z \geq 0
\end{aligned}$$

4.61 Since X and Y are independent, $E[X^2Y] = E[X^2]E[Y] = 1 \times 1 = 1$.

4.78 (a)

$$P[\sqrt{X^2+Y^2} \le r] = \int\int_{x^2+y^2\le r^2} \frac{e^{-(x^2+y^2)/2}}{2\pi} dxdy \ .$$

Let $x = rcos\theta$ and $y = rsin\theta$. Then

$$P[\sqrt{X^2+Y^2} \le r] = \int_0^{2\pi}\int_0^r \frac{e^{-r^2/2}}{2\pi} r dr d\theta = 1 - e^{-r^2/2} = 1/2 \ ,$$

which results in $r = \sqrt{2\ln(2)}$. (Note that $\sqrt{X^2+Y^2}$ has a Rayleigh distribution.)

(b) Using (a), we have $P[R \triangleq \sqrt{X^2+Y^2} > r] = e^{-r^2/2}$. Also,

$$f_{X,Y}(x,y|R>r) = \frac{f_{X,Y}(x,y)}{P[R>r]} = \frac{e^{-(x^2+y^2-r^2)/2}}{2\pi} \ .$$

4.81

$$h(x,y) = \frac{e^{-(x^2-2\rho_1 xy+y^2)/2(1-\rho_1^2)}}{2\pi\sqrt{1-\rho_1^2}} \quad , \quad g(x,y) = \frac{e^{-(x^2-2\rho_2 xy+y^2)/2(1-\rho_2^2)}}{2\pi\sqrt{1-\rho_2^2}} \ .$$

(a) $f_X(x) = \frac{1}{2}\int_{-\infty}^{\infty} h(x,y)dy + \frac{1}{2}\int_{-\infty}^{\infty} g(x,y)dy = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$. Similarly, $f_Y(y) = \frac{e^{-y^2/2}}{\sqrt{2\pi}}$. Thus, each X and Y is, individually, a Gaussian random variable.

(b) However,

$$f_{X,Y}(x,y) = \frac{\sqrt{1-\rho_2^2}e^{-(x^2-2\rho_1 xy+y^2)/2(1-\rho_1^2)} + \sqrt{1-\rho_1^2}e^{-(x^2-2\rho_2 xy+y^2)/2(1-\rho_2^2)}}{4\pi\sqrt{1-\rho_1^2}\sqrt{1-\rho_2^2}}$$

does not have the form required for <u>jointly</u> Gaussian random variables.

- **Supplementary:**

1 (a) $F_Y(y) = P[Y \le y] = P[X_1 \le y, \ldots, X_n \le y] = F_X(y)^n$.

(b) $P[Z > z] = P[X_1 > z, \ldots, X_n > z] = (1 - F_X(z))^n$. So, $F_Z(z) = 1 - P[Z > z] = 1 - (1 - F_X(z))^n$.

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2 The mapping $(y_1, y_2) = \mathbf{g}(x_1, x_2) = (x_1/x_2, x_1 x_2)$ has the inverse

$$(x_1, x_2) = \mathbf{h}(y_1, y_2) = \left((y_1 y_2)^{1/2}, (y_2/y_1)^{1/2}\right) \ .$$

We have

$$J(y_1, y_2) = \det\begin{pmatrix} \frac{1}{2}y_1^{-1/2}y_2^{1/2} & \frac{1}{2}y_1^{1/2}y_2^{-1/2} \\ -\frac{1}{2}y_1^{-3/2}y_2^{1/2} & \frac{1}{2}y_1^{-1/2}y_2^{-1/2} \end{pmatrix} = \frac{1}{2}y_1^{-1} \ .$$

Thus

$$f_{Y_1,Y_2}(y_1, y_2) = 1/2y_1 \quad \text{for} \quad 0 < y_1 y_2 < 1, \ 0 < y_2 < y_1 \ .$$

3 (a) Let X denote the amount of time (in hours) until the miner reaches safety, and let Y denote the door he initially chooses. Now

$$\begin{aligned} E[X] &= E[X|Y=1]P(Y=1) + E[X|Y=2]P(Y=2) \\ &\quad + E[X|Y=3]P(Y=3) \\ &= \frac{1}{3}(E[X|Y=1] + E[X|Y=2] + E[X|Y=3]) \end{aligned}$$

However,

$$\begin{aligned} E[X|Y=1] &= 3 \\ E[X|Y=2] &= 5 + E[X] \\ E[X|Y=3] &= 7 + E[X] \end{aligned}$$

To understand why the above equations are correct, consider, for instance, $E[X|Y=2]$ and reason as follows: If the miner chooses the second door, he spends 5 hours in the tunnel and then returns back to the mine. But once he returns to the mine the problem is exactly as it was before; thus his expected additional time until safety is just E[X]. Hence $E[X|Y=2] = 5 + E[X]$. We thus have

$$E[X] = \frac{1}{3}(3 + 5 + E[X] + 7 + E[X]) \Longrightarrow E[X] = 15\,.$$

(b)

$$\begin{aligned} f_Y(y) &= \int_{-y}^{y} \frac{1}{2} e^{-y}\, dx = ye^{-y}, \ \text{ for } \ y \geq 0\,. \\ f_X(x|Y=y) &= (1/2)e^{-y}/ye^{-y} = 1/2y, \ \text{ for } \ -y \leq x \leq y\,. \\ P(X \leq 1|Y=3) &= \int_{-3}^{1} 1/6\, dx = 2/3\,. \end{aligned}$$