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Stochastic Processes, 94.553

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Problem Set #2 Solutions

- **Textbook:** Ch. 3: 27, 37, 45, 59.

3.27 (a) Assume $x > t$, otherwise $F_X(x|X > t) = 0$. Then

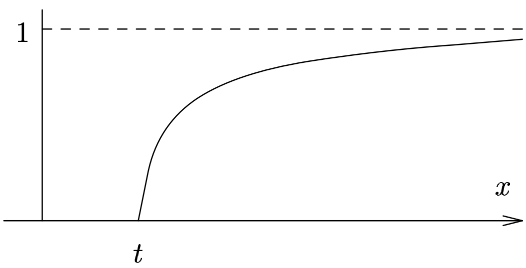
$$\begin{aligned} F_X(x|X > t) &= P[\{X \le x\} \cap \{X > t\}]/P[X > t] \\ &= P[t < X \le x]/(1 - P[X \le t]) \\ &= (F_X(x) - F_X(t))/(1 - F_X(t)) \\ &= (e^{-\lambda t} - e^{-\lambda x})/e^{-\lambda t} \\ &= 1 - e^{-\lambda(x-t)}\ , \end{aligned}$$

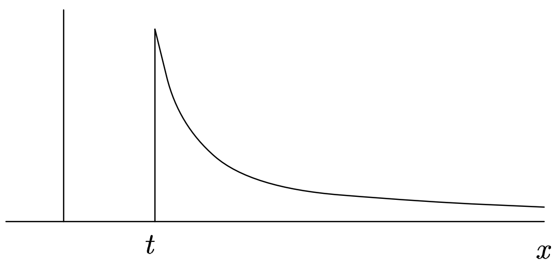
which is just a shifted version of $F_X(x)$. The plot for $F_X(x|X > t)$ is given below, on the left.

(b) For $x > t$,

$$\begin{aligned} f_X(x|X > t) &= \frac{d}{dx} F_X(x|X > t) \\ &= \lambda e^{-\lambda(x-t)}\ . \end{aligned}$$

This is plotted below on the right.





(c)

$$\begin{aligned} P[X > t + x|X > t] &= 1 - F_X(t + x|X > t) \\ &= 1 - (1 - e^{-\lambda(x+t-t)}) \\ &= 1 - (1 - e^{-\lambda x}) \\ &= 1 - F_X(x) \\ &= P[X > x] \end{aligned}$$

The probability of waiting an additional x seconds doesn't depend on the previous waiting time t. It is the same as when one begins to wait. The system, therefore, has no memory of the previous waiting; hence this is called the memoryless property.

3.37 N is a Poisson random variable with probability mass function $p_N(k) = \lambda^k e^{-\lambda}/k!$, with $\lambda = 15$.

(a) $p_N(0) = e^{-15} = 3.06 \times 10^{-7}$.

(b) $P[N > 10] = 1 - P[N \le 10] = 1 - \sum_{k=0}^{10} p_N(k) = 1 - e^{-15}(1 + 15 + 15^2/2 + 15^3/3! + \cdots + 15^{10}/10!) \approx 0.8815$.

3.45 $P[\text{error}|v = -1] = P[Y \ge 0|v = -1] = P[-1+N \ge 0] = P[N \ge 1] = \int_1^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = Q(1) = 0.159$ from Table 3.3. $P[\text{error}|v = 1] = P[Y < 0|v = 1] = P[1 + N < 0] = P[N < -1] = 1 - Q(-1) = Q(1) = 0.159$.

3.59 (a) If $y \le 0$, $P[Y \le y] = 0$. If $y > 0$, $P[Y \le y] = P[e^X \le y] = P[X \le \ln y] = F_X(\ln y)$, therefore,

$$F_Y(y) = \begin{cases} 0 & y \le 0 \\ F_X(\ln y) & y > 0 \end{cases}$$

For $y > 0$, $f_Y(y) = \frac{d}{dy} F_Y(y) = F_X'(\ln y) \frac{d}{dy} \ln y = f_X(\ln y)/y$.

(b) If X is a Gaussian random variable, then

$$f_Y(y) = \begin{cases} 0 & y \le 0 \\ \frac{e^{-(\ln y - m)^2/2\sigma^2}}{\sqrt{2\pi}\sigma y} & y > 0 \end{cases}$$

- **Supplementary:**

1 There is only one discrete point, $X = 0$, and this point has probability $1/4$. It follows that X is a mixture of two random variables, X_1 and X_2, where X_1 has a probability of one at the point zero and X_2 has the given exponential density. That is,

$$F_1(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \ge 0 \end{cases}$$

and

$$F_2(x) = \begin{cases} 0 & \text{if } x < 0 \\ \int_0^x e^{-y}\, dy = 1 - e^{-x} & \text{if } x \ge 0 \end{cases}$$

Now,

$$F(x) = \left(\frac{1}{4}\right) F_1(x) + \left(\frac{3}{4}\right) F_2(x)$$

Hence,

$$\begin{aligned} P(X > 10) &= 1 - P(X \le 10) \\ &= 1 - F(10) \\ &= 1 - \left[\frac{1}{4} + \left(\frac{3}{4}\right)(1 - e^{-10})\right] \\ &= \left(\frac{3}{4}\right)[1 - (1 - e^{-10})] = \left(\frac{3}{4}\right) e^{-10} \end{aligned}$$

2 Let $g(X)$ denote the retailer's daily profit in dollars. Then,

$$g(X) = \begin{cases} 5X, & 0 \le x \le 1 \\ 5 + 8(X-1), & 1 < x \le 2 \end{cases}$$

The expected profit for each day is then

$$\begin{aligned} E[g(X)] &= \int_{-\infty}^{\infty} g(x)f(x)dx \\ &= \int_0^1 5x\left(\frac{3}{8}x^2\right)dx + \int_1^2 (8x-3)\left(\frac{3}{8}x^2\right)dx \\ &= \frac{15}{32}\left[x^4\right]_0^1 + \frac{3}{4}\left[x^4\right]_1^2 - \frac{3}{8}[x^3]_1^2 \\ &= 9.09 \end{aligned}$$

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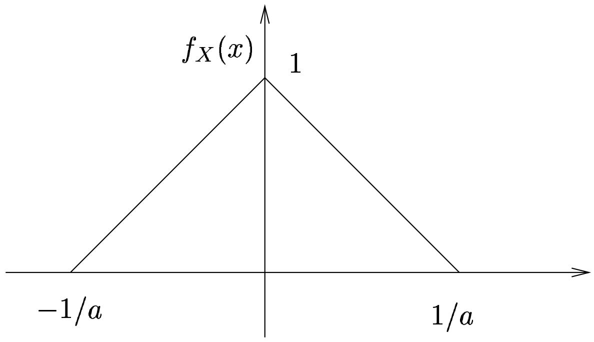
3 The pdf of X is shown in the following figure.



(a) Using $\int_{-\infty}^{\infty} f_X(x)dx = 1$, we obtain $a = 1$. We also have:

$$E(X) = 0$$

and,

$$\sigma_X = \sqrt{E(X^2)} = \sqrt{1/6}$$

(b)

$$E(Y) = E(b|X|) = b[\int_{-1}^{0}(-x)f_X(x)dx + \int_0^1 xf_X(x)dx] = 2b\int_0^1 x(1-x)dx = \frac{b}{3}$$

and, similarly,

$$E(Y^2) = 2b^2\int_0^1 x^2(1-x)dx = \frac{b^2}{6}$$

This results in $\sigma_Y = \sqrt{[E(Y^2)] - [E(Y)]^2} = \sqrt{b^2/18}$.

(c) For the case of half wave rectifier, the output is a mixed-type random variable (note that all the negative values of X are mapped to zero, i.e., the output is equal to zero with probability 1/2). We therefore have

$$E(Y) = (0 \times 1/2) + \int_0^1 bx(1-x)dx = \frac{b}{6}$$

and

$$E(Y^2) = (0^2 \times 1/2) + \int_0^1 b^2 x^2 (1-x) dx = \frac{b^2}{12}$$

This results in $\sigma_Y = \sqrt{[E(Y^2)] - [E(Y)]^2} = \sqrt{b^2/18}$.

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4 By Chebyshev inequality, we have $P(|X-m| \geq a) \leq \sigma^2/a^2$. This results in $1-P(|X-m| \leq a) \leq \sigma^2/a^2$, and therefore $P(|X-m| \leq a) \geq 1-(\sigma^2/a^2)$. The bound is nontrivial for $a > \sigma$.

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5 (a) $E(Y) = \sum_i P(X = x_i) \log_2 \frac{1}{P(X=x_i)} = 4 \times \frac{1}{8} \log_2 8 + \frac{1}{2} \log_2 2 = 3/2 + 1/2 = 2$.

(b) Let X be the number of flashes during $(0,t)$, and let A be the event that the repeater is still functioning after t seconds. Then

$$\begin{aligned} P(A) &= \sum_{k=0}^{\infty} P(A|X=k)P(X=k) \\ &= \sum_{k=0}^{\infty} p^k \frac{(\lambda t)^k}{k!} e^{-\lambda t} = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(p\lambda t)^k}{k!} = e^{-\lambda t} e^{p\lambda t} = e^{-(1-p)\lambda t}\,. \end{aligned}$$