Carleton University
Department of Systems and
Computer Engineering

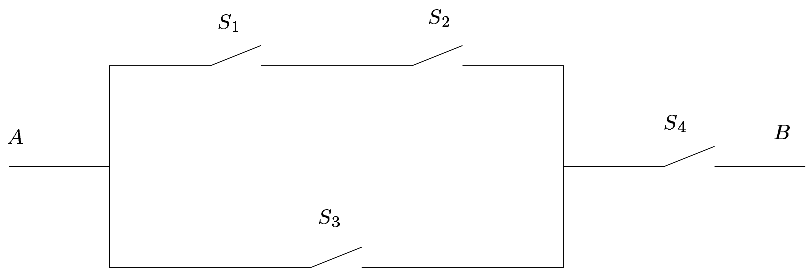
Stochastic Processes 94553

A. H. Banihashemi

Problem Set #1 Solutions

1 The maximum value of $P(A \cap B)$ is equal to $P(A) = 0.4$ and happens if $A \subset B$. The minimum value of $P(A \cap B)$ is 0.1 and happens if $A \cup B = S$.

2



E_1: Switch S_1 is closed. $\Longrightarrow$ E_1^c: Switch S_1 is open.
E_2: Switch S_2 is closed. $\Longrightarrow$ E_2^c: Switch S_2 is open.
E_3: Switch S_3 is closed. $\Longrightarrow$ E_3^c: Switch S_3 is open.
E_4: Switch S_4 is closed. $\Longrightarrow$ E_4^c: Switch S_4 is open.

<u>First method:</u>

$\mathcal{E} = [(E_1 \cap E_2) \cup E_3] \cap E_4$
$P(\mathcal{E}) = P[(E_1 \cap E_2) \cup E_3] \times P(E_4)$
$P[(E_1 \cap E_2) \cup E_3] = P(E_1 \cap E_2) + P(E_3) - P(E_1 \cap E_2 \cap E_3) = a + a^2 - a^3$
$P(\mathcal{E}) = a(a + a^2 - a^3) = a^2 + a^3 - a^4$

<u>Second method (fallacy):</u>

$P\{E_4 \cap [E_3 \cup (E_1 \cap E_2)]\} = P\{E_4 \cap [(E_3 \cup E_1) \cap (E_3 \cup E_2)]\}$
$P(E_3 \cup E_1) = P(E_3) + P(E_1) - P(E_3 \cap E_1) = a + a - a^2 = 2a - a^2$
$P(E_3 \cup E_2) = P(E_3) + P(E_2) - P(E_3 \cap E_2) = a + a - a^2 = 2a - a^2$
$P\{E_4 \cap [(E_3 \cup E_1) \cap (E_3 \cup E_2)]\} = P(E_4) \times P(E_3 \cup E_1) \times P(E_3 \cup E_2) = a(2a - a^2)^2$
It is seen that the two answers are different. Why?!

<u>Third method:</u>

Let 1 correspond to a closed switch and 0 correspond to an open switch. Using these notations, the sample space is represented as:

		S_4	S_3	S_2	S_1	
	e_1	0	0	0	0	
	e_2	0	0	0	1	
	e_3	0	0	1	0	
	e_4	0	0	1	1	
	e_5	0	1	0	0	
	e_6	0	1	0	1	
	e_7	0	1	1	0	
	e_8	0	1	1	1	
	e_9	1	0	0	0	
	e_{10}	1	0	0	1	
	e_{11}	1	0	1	0	
OK	e_{12}	1	0	1	1	$a^3(1-a)$
OK	e_{13}	1	1	0	0	$a^2(1-a)^2$
OK	e_{14}	1	1	0	1	$a^3(1-a)$
OK	e_{15}	1	1	1	0	$a^3(1-a)$
OK	e_{16}	1	1	1	1	a^4

The $e_1, e_2, \ldots, e_{16}$ are the elementary events. $P(\mathcal{E}) = P(e_{12} \cup e_{13} \cup e_{14} \cup e_{15} \cup e_{16})$.

As the elementary events are disjoint, the probabilities add. As a result,

$P(\mathcal{E}) = P(e_{12}) + P(e_{13}) + P(e_{14}) + P(e_{15}) + P(e_{16}) = a^3(1-a) + a^2(1-a)^2 + a^3(1-a) + a^3(1-a) + a^4 = a^2 + a^3 - a^4$,

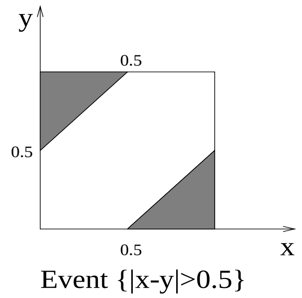
which is in agreement with the previous result.

---

3 (a) 6!. (b) 4/6. (c) 10/30. (d) $6!/(3!3!) = 20$. (e) The number of favorable cases is $2\frac{4!}{3!1!} = 8$, while the size of the sample space is 20, as computed in part (d). Thus the corresponding probability is 0.4 .

---

4 Let x and y be the two numbers selected at random. The sample space is given by the unit square shown in the Figure. The shaded region corresponds to the event $\{|x-y| > 0.5\}$. The area



Event {|x-y|>0.5}

of the shaded region is 1/4. Thus

$$P[\{|x-y| > 0.5\}] = 1/4.$$

---

5 Let H_i be the event that "a shot goes off at the ith trial". The events H_i are mutually exclusive. The event H_i occurs if there are $i-1$ "failures" and then one "success". Hence, we get (geometric probability law),

$$P(H_i) = \left(\frac{5}{6}\right)^{i-1}\frac{1}{6}$$

The probability we are looking for is given by

$$\begin{aligned} P &= P(H_1 \cup H_3 \cup H_5 \cup \cdots) = \frac{1}{6}\left[1 + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^4 + \cdots\right] \\ &= \frac{1}{6}\cdot\frac{1}{1-\left(\frac{5}{6}\right)^2} = \frac{6}{11} \end{aligned}$$

Hence the probability that B loses his life is $1 - 6/11 = 5/11$; that is, the second player has a somewhat greater chance of survival, as might be expected.

---

6 (a)

$$\begin{aligned} \epsilon_t &= P(O=1|I=0) \\ &= P(O=1|X=0, I=0)P(X=0|I=0) + P(O=1|X=1, I=0)P(X=1|I=0)\ , \end{aligned}$$

where X is the output of the first channel. We then have

$$\begin{aligned} \epsilon_t &= P(O=1|X=0)P(X=0|I=0) + P(O=1|X=1)P(X=1|I=0) \\ &= \epsilon_2(1-\epsilon_1) + (1-\epsilon_2)\epsilon_1\ . \end{aligned}$$

(b)

$$\begin{aligned} P(e) &= P(e|I=0)P(I=0) + P(e|I=1)P(I=1) \\ &= P(O=1|I=0)P(I=0) + P(O=0|I=1)P(I=1) \\ &= \epsilon_t p + \epsilon_t(1-p) = \epsilon_t \end{aligned}$$

(c) Let A be the event that at most one bit is in error. Then

$$\begin{aligned} P(A) &= P(\text{zero error}) + P(\text{one error}) \\ &= [1-P(e)]^n + \binom{n}{1}P(e)[1-P(e)]^{n-1} \\ &= (1-\epsilon_t)^n + n\epsilon_t(1-\epsilon_t)^{n-1}\ . \end{aligned}$$

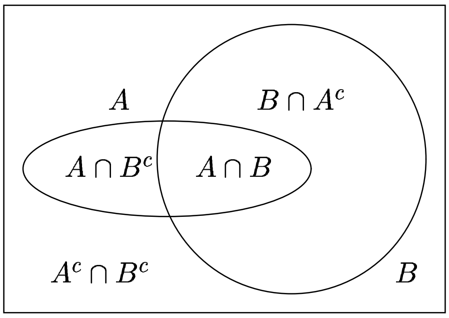
---

7 The event A is the union of the mutually exclusive events $A \cap B$ and $A \cap B^c$, thus

$$P[A] = P[A \cap B] + P[A \cap B^c]$$

so

$$\begin{aligned} P[A \cap B^c] &= P[A] - P[A \cap B] \\ &= P[A] - P[A]P[B] \quad \text{(since } A \text{ and } B \text{ are independent)} \\ &= P[A](1 - P[B]) \\ &= P[A]P[B^c]. \end{aligned}$$



Thus A and B^c are independent.

Similarly,

$$\begin{aligned} P[B \cap A^c] &= P[B] - P[A \cap B] \\ &= P[B] - P[A]P[B] \quad \text{(since } A \text{ and } B \text{ are independent)} \\ &= P[B](1 - P[A]) \\ &= P[B]P[A^c]. \end{aligned}$$

Thus B and A^c are independent.

Finally,

$$P[A^c] = P[A^c \cap B] + P[A^c \cap B^c]$$

thus

$$\begin{aligned} P[A^c \cap B^c] &= P[A^c] - P[A^c \cap B] \\ &= P[A^c] - P[A^c]P[B] \quad \text{(since } A^c \text{ and } B \text{ are independent)} \\ &= P[A^c](1 - P[B]) \\ &= P[A^c]P[B^c]. \end{aligned}$$

Thus A^c and B^c are independent.

---

8 (a) $P(e) = P(e|I = 0)P(I = 0) + P(e|I = 1)P(I = 1) = \epsilon_1 p + \epsilon_2(1 - p)$.

(b)

$$\begin{aligned} P(e) &= P(e|I = 0)p + P(e|I = 1)(1 - p) \\ &= P(\text{having more than two ones in the output block}|I = 0)p \\ &\quad + P(\text{having more than two zeros in the output block}|I = 1)(1 - p) \\ &= [\sum_{i=3}^{5} \binom{5}{i} \epsilon_1^i (1 - \epsilon_1)^{5-i}]p + [\sum_{i=3}^{5} \binom{5}{i} \epsilon_2^i (1 - \epsilon_2)^{5-i}](1 - p) \,. \end{aligned}$$

(c) For both cases, $P(e) = 0.5$. In this case, the input and the output bits are independent, and having more samples of the output bit does not help in estimating the input bit.

(d) Usinig Bayes' rule, we have

$$P(I = 1|\text{output} = 11011) = \frac{(1 - \epsilon_2)^4 \epsilon_2 (1 - p)}{\epsilon_2 (1 - \epsilon_2)^4 (1 - p) + \epsilon_1^4 (1 - \epsilon_1) p} \,.$$