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Stochastic Processes
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Problem Set #9 Solutions

- **Textbook:** Ch. 4: 93; Ch. 7: 34, 35, 39, 49, 52.

4.93

$$f_{X_1,X_2,X_3}(x_1,x_2,x_3) = \frac{\exp(-x_1^2 - x_2^2 + \sqrt{2}x_1x_2 - \frac{1}{2}x_3^2)}{2\pi\sqrt{\pi}}$$

$$= \exp\left\{\frac{-\frac{1}{s(1-p^2)}[x_1^2 - 2px_1x_2 + x_2^2]}{2\pi\sqrt{1-p^2}}\right\} \frac{\exp\left(-\frac{x_3^2}{2}\right)}{\sqrt{2\pi}}$$

X_3 is independent of X_1, X_2

$$E[X_3|X_1X_2] = E[X_3|X_2] = E[X_3] = 0$$

which is the mmse estimator for X_3.

7.34 a) If input is δ_n then $Y_0 = 1$ and $Y_1 = \frac{3}{4}$. We seek a solution to

$$Y_n = \frac{3}{4}Y_{n-1} - \frac{1}{8}Y_{n-2}$$

of the form

$$Y_n = c_1z_1^n + c_2z_2^n$$

that satisfies the above boundary conditions. The z_i must satisfy

$$cz^n = \frac{3}{4}cz^{n-1} - \frac{1}{8}cz^{n-2} \Rightarrow z^2 - \frac{3}{4}z + \frac{1}{8} = 0$$

$$\Rightarrow z_1 = \frac{1}{2} \qquad z_2 = \frac{1}{4}$$

Boundary Conditions $\Rightarrow$ $\left.\begin{array}{l} Y_0 = 1 = c_1 + c_2 \\ Y_1 = \frac{3}{4} = \frac{c_1}{2} + \frac{c_2}{4} \end{array}\right\}$ $\quad c_1 = 2$, $c_2 = -1$

$$\Rightarrow Y_n = 2\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n \qquad n \geq 0$$

b)

$$H(f) = 2\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n e^{-j2\pi fn} - \sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n e^{-j2\pi fn}$$

$$= 2\frac{1}{1-\frac{1}{2}e^{-j2\pi f}} - \frac{1}{1-\frac{1}{4}e^{-j2\pi f}}$$

$$= \frac{\frac{1}{2}e^{-j2\pi f}}{(1-\frac{1}{2}e^{-j2\pi f})(1-\frac{1}{4}e^{-j2\pi f})}$$

c)
$$S_Y(f) = |H(f)|^2\sigma_W^2$$
$$= \frac{\sigma_W^2/4}{(\frac{5}{4} - \cos 2\pi f)(\frac{17}{16} - \frac{1}{2}\cos 2\pi f)}$$
$$S_Y(f) = \frac{4}{7}\left(\frac{4}{3}\right)\frac{\frac{3}{4}\sigma_W^2}{\frac{5}{4} - \cos 2\pi f} - \frac{2}{7}\frac{16}{15}\frac{\frac{15}{16}\sigma_W^2}{\frac{17}{16} - \frac{1}{2}\cos 2\pi f}$$
$$\Rightarrow R_Y(k) = \left[\frac{16}{21}\left(\frac{1}{2}\right)^{|k|} - \frac{32}{105}\left(\frac{1}{4}\right)^{|k|}\right]\sigma_W^2$$

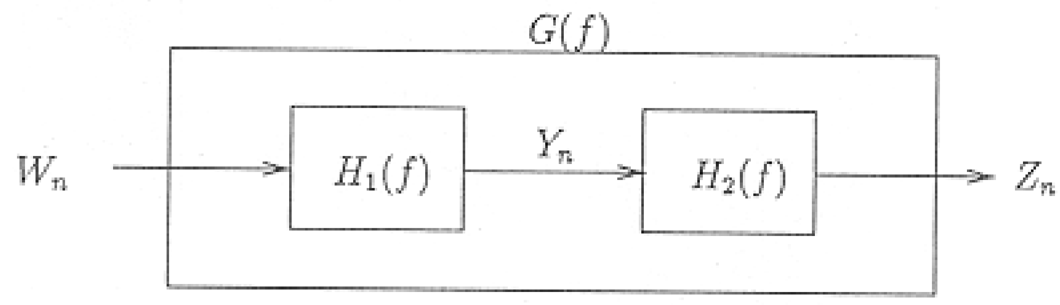
See Problem 7.9 solution.

7.35 $Z_n = Y_n - \frac{1}{4}Y_{n-1}$

a) The impulse response for this system is

$$h_0 = 1 \quad h_1 = -\frac{1}{4} \quad h_n = 0 \quad n \neq 0 \text{ or } 1$$
$$\therefore H_2(f) = 1 - \frac{1}{4}e^{-j2\pi f}$$

Let W_n be the input to the system in Problem 7.34, then



where
$$H_1(f) = \frac{\frac{1}{2}e^{-j2\pi f}}{(1 - \frac{1}{2}e^{-j2\pi f})(1 - \frac{1}{4}e^{-j2\pi f})}$$

The power spectral density of Z_n is
$$S_Z(f) = |G(f)|^2 S_W(f) = |H_1(f)H_2(f)|^2\sigma_W^2$$
$$= \frac{\sigma_W^2/4}{\frac{5}{4} - \cos 2\pi f}$$

where $G(f)$ is the transfer function that defines a first-order autoregressive process.

$$R_Z(k) = \frac{\sigma_W^2}{4}\frac{4}{3}\mathcal{F}^{-1}\left[\frac{\frac{3}{4}}{\frac{5}{4} - \cos 2\pi f}\right] = \frac{\sigma_W^2}{3}\left(\frac{1}{2}\right)^{|k|}$$

b) $G(f) = H_1(f)H_2(f) = \dfrac{\frac{1}{2}e^{-j2\pi f}}{1 - \frac{1}{2}e^{-j2\pi f}} = \dfrac{1}{2}e^{-j2\pi f}\displaystyle\sum_{\ell=0}^{\infty}\left(\frac{1}{2}e^{-j2\pi f}\right)^{\ell}$ which corresponds to a first-order autoregressive process.

c) If we let $H_3(f) = 1 - \frac{1}{2}e^{-j2\pi f}$, then

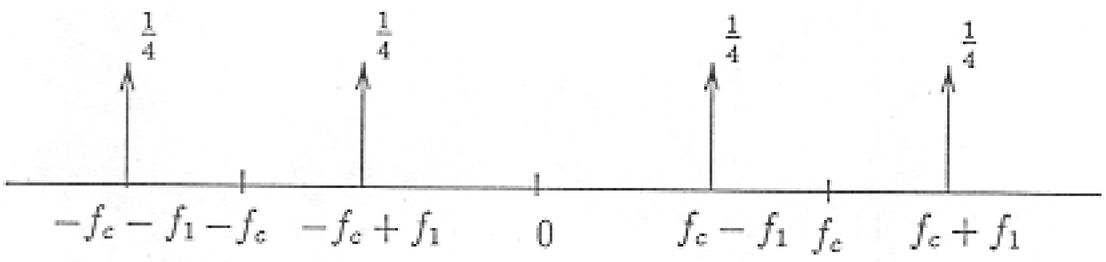
$$|H_1(f)H_2(f)H_3(f)|^2 = \frac{1}{4}$$

and

$$|H_3(f)|^2 S_Z(f) = \frac{\sigma_W^2}{4}$$

7.39

$$
\begin{aligned}
A(t) &= 2\cos(2\pi f_1 + \Phi) \\
R_A(\tau) &= \cos 2\pi f_1 \tau \qquad S_A(f) = \frac{1}{2}\delta(f + f_1) + \frac{1}{2}\delta(f - f_1) \\
S_X(f) &= \frac{1}{2}S_A(f + f_c) + \frac{1}{2}S_A(f - f_c) \\
&= \frac{1}{4}[\delta(f + f_c + f_1) + \delta(f + f_c - f_1) \\
&\quad + \delta(f - f_c + f_1) + \delta(f - f_c - f_1)]
\end{aligned}
$$



7.49 $\hat{X}(t) = aX(t_1) + bX(t_2)$

a)

$$
\begin{aligned}
e(t) &= \hat{X}(t) - X(t) \\
&= aX(t_1) + bX(t_2) - X(t)
\end{aligned}
$$

Orthogonality condition implies that

$$
\begin{aligned}
\mathcal{E}[(aX(t_1) + bX(t_2) - X(t))X(t_1)] &= 0 \\
\mathcal{E}[(aX(t_1) + bX(t_2) - X(t))X(t_2)] &= 0
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \quad aR_X(0) + bR_X(t_2 - t_1) &= R_X(t - t_1) \\
aR_X(t_1 - t_2) + bR_X(0) &= R_X(t - t_2)
\end{aligned}
$$

$$
\Rightarrow \quad \begin{bmatrix} R_X(0) & R_X(t_2 - t_1) \\ R_X(t_2 - t_1) & R_X(0) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} R_X(t - t_1) \\ R_X(t - t_2) \end{bmatrix}
$$

$$
\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{R_X^2(0) - R_X^2(t_2 - t_1)} \begin{bmatrix} R_X(0)R_X(t - t_1) - R_X(t_2 - t_1)R_X(t - t_2) \\ R_X(0)R_X(t - t_2) - R_X(t_2 - t_1)R_X(t - t_1) \end{bmatrix}
$$

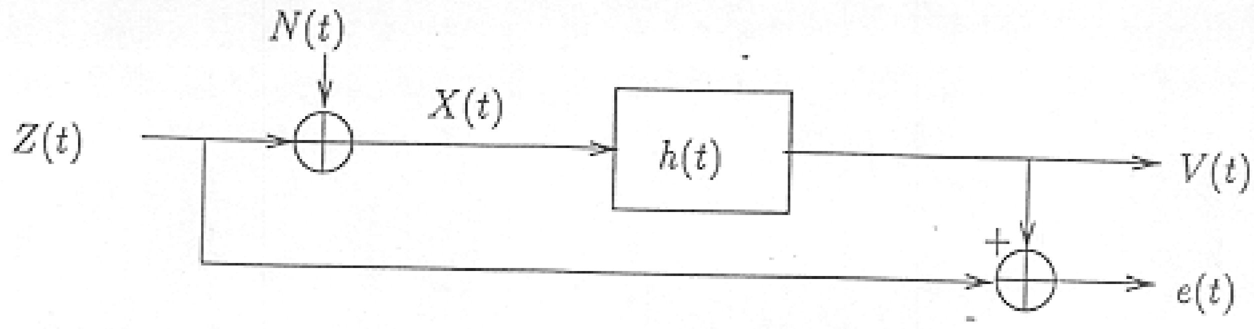
b)

$$
\begin{aligned}
\mathcal{E}[e^2(t)] &= \mathcal{E}[e(t)[aX(t_1) + bX(t_2) - X(t)]] \\
&= \underbrace{a\mathcal{E}[e(t)X(t_1)]}_{0} + \underbrace{b\mathcal{E}[e(t)X(t_2)]}_{0} - \mathcal{E}[e(t)X(t)] \\
&= -a\mathcal{E}[X(t_1)X(t)] - b\mathcal{E}[X(t_2)X(t)] + \mathcal{E}[X(t)X(t)]
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{E}[e^2(t)] = R_X(0) &- \frac{R_X(0)R_X(t - t_1) - R_X(t_2 - t_1)R_X(t - t_2)}{R_X^2(0) - R_X^2(t_2 - t_1)} R_X(t - t_1) \\
&- \frac{R_X(0)R_X(t - t_2) - R_X(t_2 - t_1)R_X(t - t_1)}{R_X^2(0) - R_X^2(t_2 - t_1)} R_X(t - t_2)
\end{aligned}
$$

Check: If $t = t_1$ then $a = 1$, $b = 0$ and $\mathcal{E}[e^2(t)] = 0$.

7.52



In Problem 7.26 we considered the above system. After making adjustments for the difference in notation, we have

$$S_e(f) = |1 - H(f)|^2 S_Z(f) + |H(f)|^2 S_N(f)$$

Equation 7.87 implies that

$$\begin{aligned}
|1 - H(f)|^2 &= \left(\frac{S_N(f)}{S_Z(f) + S_N(f)}\right)^2 \\
\therefore \quad S_e(f) &= \frac{S_N^2(f) S_Z(f)}{(S_Z(f) + S_N(f))^2} + \frac{S_N(f) S_Z^2(f)}{(S_Z(f) + S_N(f))^2} \\
&= \frac{S_N(f) S_Z(f)}{S_Z(f) + S_N(f)} \\
\mathcal{E}[e^2(t)] &= R_e(0) = \int_{-\infty}^{\infty} \frac{S_N(f) S_Z(f)}{S_Z(f) + S_N(f)} df.
\end{aligned}$$