Carleton University
Department of Systems and
Computer Engineering

Stochastic Processes
94.553 Winter 2000
A. H. Banihashemi

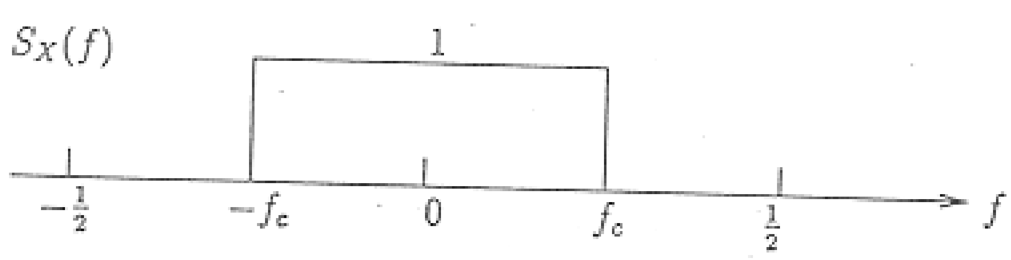
Problem Set #8 Solutions

- **Textbook:** Ch. 7: 9, 14, 15, 19, 22, 27, 30.

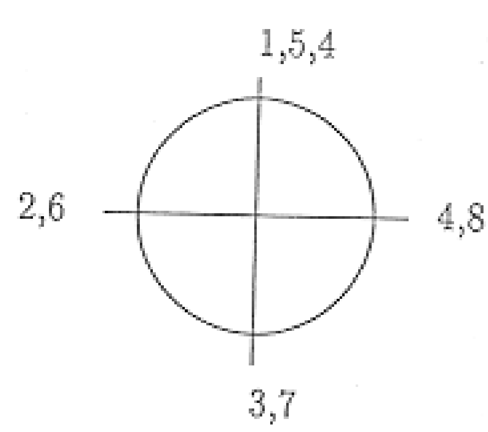
7.9

$$
\begin{aligned}
\sum_{k=-\infty}^{\infty} \alpha^{|k|} e^{-j2\pi fk} &= 1+\sum_{k=1}^{\infty} \alpha^k e^{-j2\pi fk} + \sum_{k=-\infty}^{-1}\left(\frac{1}{\alpha}\right)^k e^{-j2\pi fk} \\
&= 1+\frac{\alpha e^{-j2\pi f}}{1-\alpha e^{-j2\pi f}}+\frac{\alpha e^{j2\pi f}}{1-\alpha e^{j2\pi f}} \\
&= \frac{1-\alpha^2}{1+\alpha^2-2\alpha\cos 2\pi f} \\
S_X(f) &= \mathcal{F}\left[4\left(\frac{1}{2}\right)^{|k|}+16\left(\frac{1}{4}\right)^{|k|}\right] \\
&= 4\frac{1-\frac{1}{4}}{1+\left(\frac{1}{2}\right)^2-2\left(\frac{1}{2}\right)\cos 2\pi f}+16\frac{1-\left(\frac{1}{4}\right)^2}{1+\left(\frac{1}{4}\right)^2-2\left(\frac{1}{4}\right)\cos 2\pi f} \\
&= \frac{12}{5-4\cos 2\pi f}+\frac{240}{17-8\cos 2\pi f}
\end{aligned}
$$

7.14



$$
\begin{aligned}
R_X(k) &= \int_{-f_c}^{f_c} e^{j2\pi fk}df = \left.\frac{e^{j2\pi fk}}{j2\pi k}\right|_{-f_c}^{f_c} = \frac{e^{j2\pi f_c k}-e^{-j2\pi f_c k}}{j2\pi k} \\
&= \frac{\sin 2\pi f_c k}{\pi k} = 2f_c \operatorname{Sinc}(2f_c k)
\end{aligned}
$$



If $f_c = \frac{1}{4}$,

$$
R_X(k) = \frac{\sin k\frac{\pi}{2}}{\pi k} = \begin{cases} \frac{1}{2} & k=0 \\ \frac{1}{\pi k} & k=1,5,9,\ldots \\ -\frac{1}{\pi k} & k=3,7,11,\ldots \\ 0 & k \text{ even} \end{cases}
$$

7.15 a) $\mathcal{E}[Y_n] = \mathcal{E}[W_n]\mathcal{E}[X_n] = 0$

$$\begin{aligned}
\mathcal{E}[Y_n, Y_{n+k}] &= \mathcal{E}[W_n W_{n+k} X_n X_{n+k}] \\
&= \mathcal{E}[W_n W_{n+k}]\mathcal{E}[X_n X_{n+k}] \\
&= \delta_{k,0}\mathcal{E}[X_n^2] \Rightarrow Y_n \text{ is a white noise sequence}
\end{aligned}$$

assuming $R_W(0) = 1$ $\quad$ X_n WSS

$$\sigma_{Y_n}^2 = R_Y(0) = \mathcal{E}[X_n^2]$$

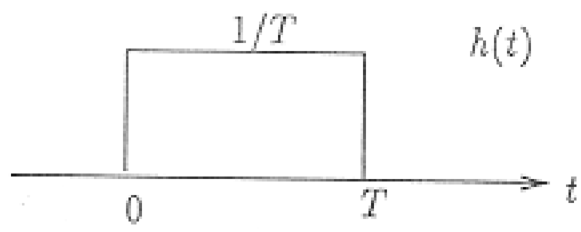
b)

$$\begin{aligned}
\mathcal{E}[Y_n] &= 0 \\
R_Y(k) &= \begin{cases} \left(\frac{1}{2}\right)^0 = 1 & k = 0 \\ 0 & \end{cases}
\end{aligned}$$

there is not, however, enough information to find joint pdf's of Y_n.

7.19 The impulse response is

$$\begin{aligned}
h(t) &= \frac{1}{T}\int_{t-T}^{t} \delta(t')dt' = \frac{1}{T}\int_{-\infty}^{t} \delta(t')dt' - \frac{1}{T}\int_{-\infty}^{t-T} \delta(t')dt' \\
&= \frac{1}{T}[u(t) - u(t-T)]
\end{aligned}$$



$$\begin{aligned}
H(f) &= \frac{1}{T}\int_0^T e^{-j2\pi ft}dt = \frac{1}{T}\frac{1 - e^{-j2\pi ft}}{j2\pi f} = \frac{1}{T}\frac{e^{j2\pi f\frac{T}{2}} - e^{-j2\pi f\frac{T}{2}}}{j2\pi f}e^{-j2\pi f\frac{T}{2}} \\
&= \frac{1}{T}\frac{\sin \pi fT}{\pi f}e^{-j\pi fT}
\end{aligned}$$

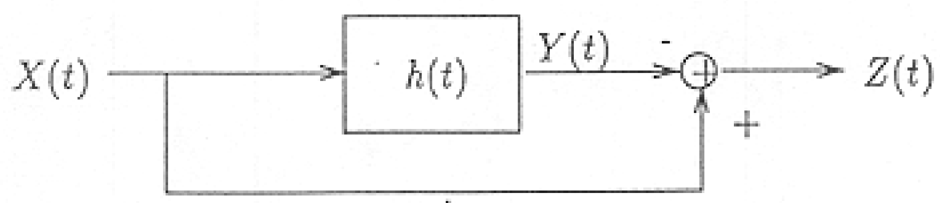
$$\begin{aligned}
S_Y(f) &= |H(f)|^2 S_X(f) \\
&= \frac{\sin^2 \pi fT}{T^2\pi^2 f^2}S_X(f)
\end{aligned}$$

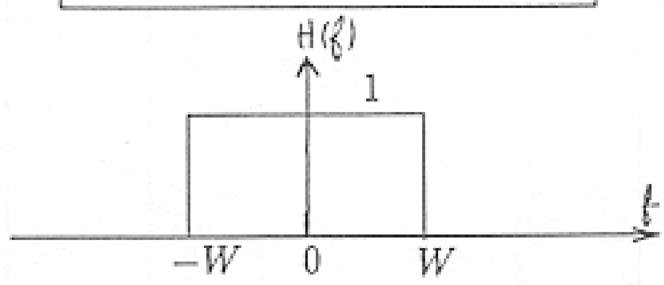
7.22

$$\begin{aligned}
Y(t) &= \int_0^{\infty} h(t-s)X(s)ds = \int_{-\infty}^{t} h(u)X(t-u)du \\
\mathcal{E}[Y(t)] &= \int_{-\infty}^{t} h(u)\mathcal{E}[X(t-u)]du = m_X \int_{-\infty}^{t} h(u)du \\
\mathcal{E}[Y(t)Y(t+\tau)] &= \mathcal{E}\left[\int_{-\infty}^{t}\int_{-\infty}^{t+\tau} h(u)h(v)X(t-u)X(t+\tau-v)dudv\right] \\
&= \int_{-\infty}^{t}\int_{-\infty}^{t+\tau} h(u)h(v)R_X(\tau+u-v)dudv
\end{aligned}$$

depends on t and $t+\tau$. As $t \to \infty$ this expression approaches the expression in Eqn. 7.42 and the process becomes WSS.

7.27





In general, if $Z(t) = X(t) - y(t)$ and $y(t) = h(t) * X(t)$, we have

$R_Z(\tau) = E[\{X(t) - y(t)\}\{X^*(t-\tau) - y^*(t-\tau)\}] = R_X(\tau) - R_{Xy}(\tau) - R_{yX}(\tau) + R_y(\tau)$.

Thus,

$$\begin{aligned}
S_Z(f) &= S_X(f) - S_X(f)H^*(f) - S_X(f)H(f) + |H(f)|^2 S_X(f) \\
&= |1-H(f)|^2 S_X(f)
\end{aligned}$$

For this particular problem,

$$\begin{aligned}
S_Z(f) &= |1-H(f)|^2 S_X(f) \\
&= \begin{cases} 0 & |f| < W \\ \frac{4\alpha}{4\alpha^2+4\pi^2 f^2} & |f| > W \end{cases} \\
\mathcal{E}[Z^2(t)] &= 2\int_W^{\infty} \frac{4\alpha}{4\alpha^2+4\pi^2 f^2} df \qquad \begin{matrix} x = 2\pi f \\ dx = 2\pi df \end{matrix} \\
&= 8\alpha \int_{2\pi W}^{\infty} \frac{1}{4\alpha^2 + x^2}\frac{dx}{2\pi} \\
&= \frac{4\alpha}{\pi}\frac{1}{2\alpha}\tan^{-1}\frac{x}{2\alpha}\bigg|_{2\pi W}^{\infty} \\
&= \frac{2}{\pi}\left[\frac{\pi}{2} - \tan^{-1}\frac{\pi W}{\alpha}\right] \\
&= 1 - \frac{2}{\pi}\tan^{-1}\frac{\pi W}{\alpha}
\end{aligned}$$

7.30 a)

$$\begin{aligned} R_{YX}(k) &= \mathcal{E}[Y_{n+k}X_n] = \mathcal{E}[(X_{n+k} + \beta X_{n+k-1})X_n] \\ &= R_X(k) + \beta R_X(k-1) \\ S_{YX}(f) &= \mathcal{F}[R_{YX}(k)] = S_X(f) + \beta S_X(f)e^{-j2\pi f} \\ &= \frac{(1+\beta e^{-j2\pi f})(1-\alpha^2)}{1+\alpha^2 - 2\alpha\cos 2\pi f}\sigma^2 \end{aligned}$$

since

$$\mathcal{F}[\alpha^{|k|}] = \frac{1-\alpha^2}{1+\alpha^2 - 2\alpha\cos 2\pi f}$$

See Problem 7.9.

b)

$$\begin{aligned} R_Y(k) &= \mathcal{E}[(X_{n+k} + \beta X_{n+k-1})(X_n + \beta X_{n-1})] \\ &= (1+\beta^2)R_X(k) + \beta R_X(k+1) + \beta R_X(k-1) \\ S_Y(f) &= (1+\beta^2)S_X(f) + \beta S_X(f)e^{j2\pi f} + \beta S_X(f)e^{-j2\pi f} \\ &= [(1+\beta^2) + 2\beta\cos 2\pi f]S_X(f) \\ &= \frac{1+\beta^2+2\beta\cos 2\pi f}{1+\alpha^2-2\alpha\cos 2\pi f}(1-\alpha^2)\sigma^2 \\ \mathcal{E}[Y_n^2] &= R_Y(0) = (1+\beta^2)R_X(0) + \beta R_X(1) + \beta R_X(-1) \\ &= (1+\beta^2)\sigma^2 + 2\beta\sigma^2\alpha \end{aligned}$$

c) if $\beta = -\alpha$ then $S_Y(f) = (1-\alpha^2)\sigma^2$ and $\mathcal{E}[Y_n^2] = (1-\alpha^2)\sigma^2$.