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Stochastic Processes
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Problem Set #7 Solutions

- **Textbook:** Ch. 6: 91, 101; Ch. 7: 4,8.

6.91 Assume $X(t)$ is a WSS mean square periodic process. then

$$K_X(t_1, t_2) = K_X(t_2 - t_1)$$

Eigenvalue equation:

$$\int_0^T K_X(t_1 - t_2)\phi_k(t_2)dt_2 = \lambda_k \phi_k(t_1).$$

Try

$$\phi_k(t) = \frac{1}{\sqrt{T}} e^{j\frac{2\pi}{T}kt}.$$

$\{\phi_k(t)\}$ is orthonormal set and

$$\begin{aligned}
\int_0^T K_X(t_1 - t_2)\frac{1}{\sqrt{T}} e^{j\frac{2\pi}{T}kt_2}dt_2 &= \int_{-t_1}^{T-t_1} K_X(u)\frac{1}{\sqrt{T}} e^{j\frac{2\pi}{T}ku} e^{+j\frac{2\pi}{T}kt_1} du \\
&= \frac{1}{\sqrt{T}} e^{j\frac{2\pi}{T}kt_1} \int_0^T K_X(u) e^{j\frac{2\pi}{T}ku} du \\
&= \frac{1}{\sqrt{T}} e^{j\frac{2\pi}{T}kt_1} \cdot \lambda_k \\
&= \lambda_k \phi_k(t_1)
\end{aligned}$$

where the eigenvalue is given by

$$\lambda_k = \int_0^T K_X(u) e^{j\frac{2\pi}{T}ku} du \; .$$

Therefore the KL Expansion is:

$$X(t) = \sum_{k=-\infty}^{+\infty} X_k \phi_k(t)$$

where

$$\begin{aligned}
X_k &= \int_0^T X(t)\phi_k^*(t)dt \\
&= \int_0^T X(t)\frac{1}{\sqrt{T}} e^{-j\frac{2\pi}{T}kt} dt
\end{aligned}$$

Thus

$$\frac{X_K}{\sqrt{T}} = \frac{1}{T}\int_0^T X(t) e^{-j\frac{2\pi}{T}kt} dt, \quad \text{the Fourier coefficients}$$

Therefore $K - L$ expansion of $X(t)$ yields the Fourier series.

6.101 a) $Y_{4n+1} = A_{2n+1}$, $Y_{4n+2} = A_{2n+2}$, $Y_{4n+3} = B_{2n+1}$, $Y_{4n+4} = B_{2n+2}$, $n = 0, 1, \ldots$

$$\begin{aligned}
E[Y_{4j+k}Y_{4m+n}] &= E[A_{2j+k}A_{2m+n}] = \sigma_1^2\rho_1^{|2m+n-2j-k|} & 1 \le k,\ n \le 2 \\
E[Y_{4j+k}Y_{4m+n}] &= E[B_{2j+k-2}B_{2m+n-2}] = \sigma_2^2\rho_2^{|2m+n-2j-k|} & 3 \le k,\ n \le 4 \\
E[Y_{4j+k}Y_{4m+n}] &= 0 \quad \text{otherwise}
\end{aligned}$$

b) Y_m is not WS stationary, but is cyclostationary.

c) $m = 4n + 1$

$$f_{Y_mY_{m+1}}(y_m, y_{m+1}) = f_{A_{2n+1}A_{2n+2}}(y_m, y_{m+1}) \sim N(0, 0, \sigma_1^2, \sigma_1^2, \underbrace{\rho_1\sigma_1^2}_{\text{cov}})$$

$m = 4n + 3$

$$f_{Y_mY_{m+1}}(y_m, y_{m+1}) = f_{B_{2n+1}B_{2n+2}}(y_m, y_{m+1}) \sim N(0, 0, \sigma_2^2, \sigma_2^2, \rho_2\sigma_2^2)$$

$m = 4n + 2$

$$f_{Y_mY_{m+1}}(y_m, y_{m+1}) = f_{A_{2n+2}B_{2n+1}}(y_m, y_{m+1}) \sim N(0, 0, \sigma_1^2, \sigma_2^2, 0)$$

$m = 4n + 4$

$$f_{Y_mY_{m+1}}(y_m, y_{m+1}) = f_{B_{2n+2}A_{2n+3}}(y_m, y_{m+1}) \sim N(0, 0, \sigma_2^2, \sigma_1^2, 0)$$

d) $Z_m = Y_{m+T}$, will "stationarize" Y_m.

$$\begin{aligned}
E[Z_mZ_n] &= E[Y_{m+T}Y_{n+T}] \\
&= \sum_{T=0}^{3} E[Y_{m+T}Y_{n+T}] \cdot \frac{1}{4} = \begin{cases} \dfrac{\rho_1^{\lceil \frac{k+1}{2} \rceil}\sigma_1^2 + \rho_2^{\lceil \frac{k+1}{2} \rceil}\sigma_2^2}{4}, & |m-n| = 2k+1,\ k = 0, 1, 2, \ldots \\ 0, & |m-n| = 4k+2,\ k = 0, 1, \ldots \\ \dfrac{\rho_1^{2k}\sigma_1^2 + \rho_2^{2k}\sigma_2^2}{2}, & |m-n| = 4k,\ k = 0, 1, 2, \ldots \end{cases}
\end{aligned}$$

Z_m is stationary.

$$f_{Z_mZ_{m+1}}(z_m, z_{m+1}) = \sum_{T=0}^{3} f_{Y_{m+T}Y_{m+1+T}}(z_m, z_{m+1}) \cdot \frac{1}{4}$$

this can be computed using the results of part c.

7.4 $S_X(f) = Ap\left(\frac{f}{f_2}\right) + (B - A)p\left(\frac{f}{f_1}\right)$ where $p(x)$ is as in Fig. P7.2.
Then from Table C in Appendix B:

$$R_X(\tau) = 2Af_2\underbrace{\frac{\sin 2\pi f_2\tau}{2\pi f_2\tau}}_{\text{Sinc}(2f_2\tau)} + 2(B - A)f_1\underbrace{\frac{\sin 2\pi f_1\tau}{2\pi f_1\tau}}_{\text{Sinc}(2f_1\tau)}$$

7.8

$$\begin{aligned}
\mathcal{E}[Z(t)] &= \mathcal{E}[X(t)]\mathcal{E}[Y(t)] = m_Xm_Y \\
R_Z(t, t - \tau) &= \mathcal{E}[X(t)Y(t)X^*(t - \tau)Y^*(t - \tau)] \\
&= \mathcal{E}[X(t)X^*(t - \tau)]\mathcal{E}[Y(t)Y^*(t - \tau)] \\
&= R_X(\tau)R_Y(\tau) \\
S_Z(f) &= \mathcal{F}[R_X(\tau)R_Y(\tau)] = S_X(f) * S_Y(f)
\end{aligned}$$

Supplementary:

1. We condition on the number of occurrences $N(t)$, up to t

$$
\begin{aligned}
E[X(t)] &= E[E[X(t)|N(t)]] \\
E[X(t)|N(t)=k] &= E\left[\sum_{j=1}^{k} A_j h(t-s_j)\right] \\
&= \sum_{j=1}^{k} E[A_j]E[h(t-s_j)] \\
E[h(t-s_j)] &= \int_0^t h(t-s)\cdot\frac{1}{t}ds \\
&= \int_0^t h(u)\frac{1}{t}du \\
E[X(t)|N(t)=k] &= E[A_j]\cdot k\cdot\int_0^t h(u)\frac{1}{t}du \\
E[X(t)] &= E[E[X(t)|N(t)]] \\
&= E\left[E[A_j]\frac{N(t)}{t}\int_0^t h(u)du\right] \\
&= E[A_j]\lambda\int_0^t h(u)du
\end{aligned}
$$