Carleton University
Department of Systems and
Computer Engineering

Stochastic Processes
94.553 Winter 2000
A. H. Banihashemi

Problem Set #6 Solutions

- **Textbook:** Ch. 6: 53, 57, 61 (a), (b), 66, 71, 79.

6.53(a)

$$E[X(t)] = E[A\cos wt + B\sin wt]$$
$$= E[A]\cos wt + E[B]\sin wt = 0$$

$$C_X(t_1,t_2) = E[(A\cos wt_1 + B\sin wt_1)(A\cos wt_2 + B\sin wt_2)]$$
$$= E[A^2]\cos wt_1 \cos wt_2 + E[B^2]\sin wt_1 \sin wt_2$$
$$+E[A]\,E[B]\cos wt_1 \sin wt_2 + E[A]\,E[B]\sin wt_1 \cos wt_2$$
$$= E[A^2]\cos wt_1 \cos wt_2 + E[B^2]\sin wt_1 \sin wt_2\ .$$

Since $E[A^2] = E[B^2]$,

$$C_X(t_1,t_2) = \qquad E[A^2]\,\cos w(t_1 - t_2)\ ,$$

and thus, $X(t)$ is WSS.

(b)

$$E[X^3(t)] = E[(A\cos wt + B\sin wt)^3]$$
$$= E[A^3\cos^3 wt + 3A^2B\cos^2 wt\,\sin wt + 3AB^2\cos wt\sin^2 wt + B^3\sin^3 t]$$
$$= E[A^3]\cos^3 wt + E[B^3]\sin^3 wt$$
$$= E[A^3](\cos^3 wt + \sin^3 wt)$$

Therefore the moment of $X(t)$ depends on time and $X(t)$ is not stationary. (assuming $E(A^3) \neq 0$.)

6.57(a)

$$Z(t) = aX(t) + bY(t)$$
$$E[Z(t)] = aE[X(t)] + bE[Y(t)] = 0$$

$$C_Z(t_1, t_2) = E[Z(t_1)Z(t_2)] = E[(aX(t_1) + bY(t_1))(aX(t_2) + bY(t_2))]$$

$$= a^2E[X(t_1)X(t_2)] + abE[X(t_1)Y(t_2)] + abE[X(t_2)Y(t_1)] + b^2E[Y(t_1)Y(t_2)]$$

$= (a^2 + b^2)\, C_X(t_2 - t_1)$, since $E[X(t_1)Y(t_2)] = E[X(t_2)Y(t_1)] = 0$. Thus, $Z(t)$ is WSS.

(b) Since $X(t)$ and $Y(t)$ are independent Gaussian RV's, $Z(t)$ is a Gaussian RV with mean 0 and variance $(a^2 + b^2)\, C_X(0)$:

$$f_{Z(t)}(x) = \frac{e^{-x^2}/2(a^2 + b^2)C_X(0)}{\sqrt{2\pi(a^2 + b^2)\, C_X(0)}}$$

6.61 The sequence U_n and the sequences X_n, Y_n, and Z_n are related as shown below:

...	U_{-2}	U_{-1}	U_0	U_1	U_2	U_3	U_4	U_5	U_6	...
...	Y_{-1}	Z_{-1}	X_0	Y_0	Z_0	X_1	Y_1	Z_1	X_2	...

a)

If the sequence U_n is shifted by a multiple of 3, say be $3k$, then the subsequences X_n, Y_n and Z_n are shifted by k. If the subsequences are stationary, then the shifted U_n has the same joint distributions and thus U_n is cyclostationary.

On the other hand, if the shift of U_n is not a multiple of 3 then the joint distributions are not the same, and thus U_n is not stationary.

b) Similar to part (a).

6.66

$$\begin{aligned}
m_X(t) &= E[A] \cos \frac{2\pi t}{T} \\
E[X_S(t)] &= \frac{1}{T}\int_0^T E[A] \cos \frac{2\pi t}{T} dt = 0 \\
R_X(t, t+\tau) &= E\left[A \cos \frac{2\pi t}{T} A \cos \frac{2\pi(t+\tau)}{T}\right] \\
&= E[A^2]\frac{1}{2}\left(\cos \frac{2\pi\tau}{T} + \cos \frac{4\pi t + 2\pi\tau}{T}\right) \\
R_{X_S}(\tau) &= \frac{1}{T}\int_0^T R_X(t, t+\tau) dt \\
&= \frac{1}{2} \cos \frac{2\pi\tau}{T} E[A^2]
\end{aligned}$$

6.71 $R_X(\tau) = \sigma^2 e^{-\alpha\tau^2}$, $R_X(t_1, t_2) = \sigma^2 e^{-\alpha(t_1-t_2)^2}$

a) Yes, since $R_X(\tau)$ is continuous at $\tau = 0$ (every τ).

b) Yes, since $R_X(\tau)$ has derivatives of all orders at $\tau = 0$.

$$\begin{aligned} E\left[\frac{d}{dt}X(t)\right] &= \frac{d}{dt}[E[X(t)]] = 0, \qquad \text{because } E[X(t)] = \text{constant} \\ R_{X'}(\tau) &= -\frac{d}{d\tau^2}R_X(\tau) \\ &= 2\alpha\sigma^2 e^{-\alpha\tau^2}(1 - 2\alpha\tau^2) \end{aligned}$$

c) Yes since $R_X(\tau)$ is M.S. continuous.
Consider $Y(t) = \int_0^t X(t)dt$.

$$\begin{aligned} E[Y(t)] &= \int_0^t E[X(t)]dt = m_X t \\ R_Y(t_1, t_2) &= \int_0^{t_1}\int_0^{t_2} R_X(u, v)dudv \\ &= \int_{u=0}^{t_1}\int_{v=0}^{t_2} R_X(u-v)dudv. \end{aligned}$$

By changing variables $t = u$, $\tau = u - v$, we obtain

$$R_Y(t_1, t_2) = \int_{t=0}^{t_1}\int_{\tau=t-t_2}^{t} R_X(\tau)\, d\tau\, dt.$$

d) The fact that the autocorrelation has the shape of a Gaussian pdf does not imply that the process is Gaussian.

6.79 $R_x(\tau) = A(1 - |\tau|)$, $|\tau| \le 1$. We have

$$\text{VAR}[< X(t) >_T] = \frac{1}{2T}\int_{-2T}^{2T}\left(1 - \frac{|u|}{2T}\right) C_x(u)\, du \le \frac{1}{2T}\int_{-2T}^{2T}\left(1 - \frac{|u|}{2T}\right) R_x(u)\, du$$

$$< \frac{1}{2T}\int_{-2T}^{2T} R_x(u)\, du = \frac{1}{2T}\int_{-1}^{1} A(1 - |u|)\, du = \frac{A}{2T}, \quad \text{for } T > 1/2.$$

Therefore $\text{VAR}[< X(t) >_T] \to 0$ as $T \to \infty$, and $X(t)$ is mean-ergodic.