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Stochastic Processes
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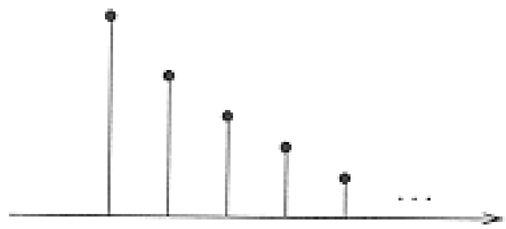
Problem Set #5 Solutions

- **Textbook:** Ch. 6: 4, 7, 16, 20, 31, 34, 35, 40, 48.

6.4 $X_n = s^n$, where $0 < s < 1$. A typical sample function looks like this:



(b) $F_{X_n}(x) = P[X_n \leq x] = P[s^n \leq x] = P[s \leq x^{1/n}] = x^{1/n}$, $0 \leq x \leq 1$. $F_{X_n}(x) = 1$, $x > 1$, and $F_{X_n}(x) = 0$, elsewhere.

(c) For $0 < x < 1, y > 0$, and $x > 0, 0 < y < 1$,

$$\begin{aligned} P[X_n \leq x, X_{n+1} \leq y] &= P[s^n \leq x, s^{n+1} \leq y] \\ &= P[s \leq x^{1/n}, s \leq y^{1/(n+1)}] \\ &= P[s \leq \min(x^{1/n}, y^{1/(n+1)})] \\ &= \min(x^{1/n}, y^{1/(n+1)}) \end{aligned}$$

It can be then seen that for $x > 0, y > 0$, $F_{X_n, X_{n+1}}(x, y) = \min(1, x^{1/n}, y^{1/(n+1)})$, and $= 0$, otherwise.

(d) $m_X(n) = E[X_n] = E[s^n] = \int_0^1 s^n ds = 1/(n+1)$.
$R_X(n, n+k) = E[X_n X_{n+k}] = E[s^n s^{n+k}] = E[s^{2n+k}] = 1/(2n+k+1)$, for $n \geq 0$, and $n+k \geq 0$.
$C_X(n, n+k) = R_X(n, n_k) - E(X_n)E(X_{n+k}) = 1/(2n+k+1) - 1/((n+1)(n+k+1)) = n(k+n)/[(n+1)(n+k+1)(2n+k+1)]$, for $n \geq 0$, and $n+k \geq 0$.

6.7 a) We will use conditional probability:

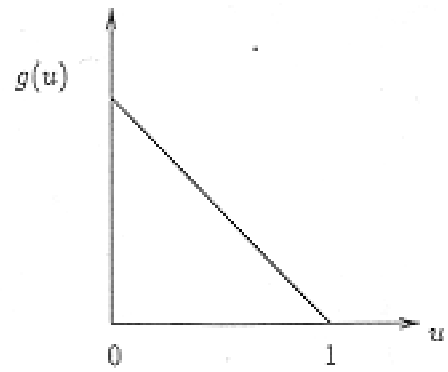
$$\begin{aligned} P[X(t) \leq x] &= P[g(t-T) \leq x] \\ &= \int_0^1 P[g(t-T) \leq x | T = \lambda] f_T(\lambda) d\lambda \\ &= \int_0^1 P[g(t-\lambda) \leq x] d\lambda \quad \text{since } f_T(\lambda) = 1 \\ &= \int_{t-1}^{t} P[g(u) \leq x] du \quad \text{after letting } u = t - \lambda \end{aligned}$$

$g(u)$ (and hence $P[g(u) \leq x]$) is a periodic function of u with period 1, so we can change the limits of the above integral to any full period. Thus

$$P[X(t) \leq x] = \int_0^1 P[g(u) \leq x] du$$

Note that $g(u)$ is deterministic, so

$$P[g(u) \leq x] = \begin{cases} 1 & u : g(u) \leq x \\ 0 & u : g(u) > x \end{cases}$$



So finally

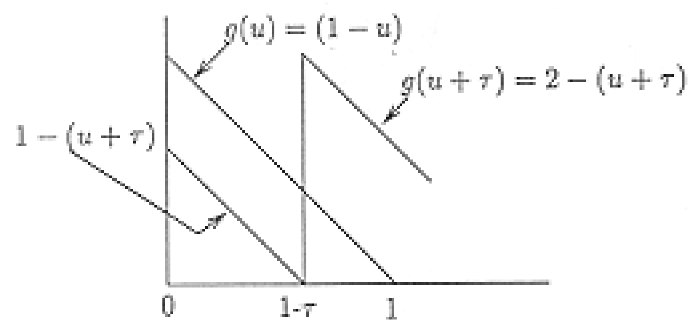
$$P[X(t) \le x] = \int_{\substack{u: g(u) \le x \\ \& \, u \in [0,1]}} 1 \, du = \int_{1-x}^{1} 1 \, du = x \, .$$

b) $m_X(t) = E[X(t)] = \int_0^1 E\left[g(t-T) \mid T=\lambda\right] d\lambda = \int_0^1 g(t-\lambda) d\lambda = \int_0^1 g(t) dt = \int_0^1 (1-t) dt = \frac{1}{2}$.
The correlation is again found using conditioning on T:

$$\begin{aligned} E[X(t)X(t+\tau)] &= \int_0^1 E[g(t-T)g(t+\tau-T) | T=\lambda] f_T(\lambda) d\lambda \\ &= \int_0^1 g(t-\lambda) g(t+\tau-\lambda) d\lambda \\ &= \int_{t-1}^{t} g(u) g(u+\tau) du \end{aligned}$$

$g(u)g(u+\tau)$ is a periodic function in u so we can change the limits to (0,1):

$$E[X(t)X(t+\tau)] = \int_0^1 g(u) g(u+\tau) du$$



here we assume $0 < \tau < 1$ since $E[X(t)X(t+\tau)]$ is periodic in τ.

$$\begin{aligned} E[X(t)X(t+\tau)] &= \int_0^{1-\tau} (1-u)(1-u-\tau) du + \int_{1-\tau}^{1} (1-u)(2-u-\tau) du \\ &= \frac{1}{3} - \frac{\tau}{2} + \frac{\tau^3}{6} + \frac{\tau^2}{2} - \frac{\tau^3}{6} \\ &= \frac{1}{3} - \frac{\tau}{2} + \frac{\tau^2}{2} \, . \end{aligned}$$

Thus

$$\begin{aligned} C_X(t, t+\tau) &= \frac{1}{3} - \frac{\tau}{2} + \frac{\tau^2}{2} - \frac{1}{4} \\ &= \frac{1}{12} - \frac{\tau}{2} + \frac{\tau^2}{2} \, , \quad 0 \leq \tau < 1, \end{aligned}$$

and $C_X(\tau) = C_X(\tau + 1)$, $\forall \tau$.

6.16 a) $\mathcal{E}[X(t)] = \mathcal{E}[A\cos\omega t + B\sin\omega t] = \mathcal{E}[A]\cos\omega t + \mathcal{E}[B]\sin\omega t = 0$

$$\begin{aligned}
C_X(t_1,t_2) &= \mathcal{E}[X(t_1)X(t_2)] - m_X(t_1)m_X(t_2) \\
&= \mathcal{E}[(A\cos\omega t_1 + B\sin\omega t_1)(A\cos\omega t_2 + B\sin\omega t_2)] \\
&= \mathcal{E}[A^2]\cos\omega t_1\cos\omega t_2 + \mathcal{E}[A]\mathcal{E}[B]\cos\omega t_1\sin\omega t_2 \\
&\quad + \mathcal{E}[A]\mathcal{E}[B]\cos\omega t_2\sin\omega t_1 + \mathcal{E}[B^2]\sin\omega t_2\sin\omega t_2 \\
&= \sigma^2(\cos\omega t_1\cos\omega t_2 + \sin\omega t_1\sin\omega t_2) \\
&= \sigma^2\cos\omega(t_1 - t_2)
\end{aligned}$$

b) Because A and B are jointly Gaussian RV's, $X(t) = A\cos\omega t + B\sin\omega t$ and $X(t+s) = A\cos\omega(t+s) + B\sin\omega(t+s)$ are also jointly Gaussian, with zero means and covariance matrix

$$K = \begin{bmatrix} \sigma^2 & \sigma^2\cos\omega s \\ \sigma^2\cos\omega s & \sigma^2 \end{bmatrix} \qquad |K|^{1/2} = \sigma^2|1-\cos^2\omega s|^{1/2} = \sigma^2|\sin\omega s|$$

$$K^{-1} = \frac{1}{\sigma^4\sin^2\omega s}\begin{bmatrix} \sigma^2 & -\sigma^2\cos\omega s \\ -\sigma^2\cos\omega s & \sigma^2 \end{bmatrix}$$

$$\begin{aligned}
f_{X(t)X(t+s)}(x_1,x_2) &= \frac{\exp\left\{-\frac{1}{2}\underline{x}^T K^{-1}\underline{x}\right\}}{2\pi\sigma^2|\sin\omega s|} \\
&= \frac{\exp\left\{-\frac{x_1^2 - 2\cos\omega s\, x_1 x_2 + x_2^2}{2\sigma^2\sin^2\omega s}\right\}}{2\pi\sigma^2|\sin\omega s|}
\end{aligned}$$

6.20

$$\begin{aligned}
P[Y_n = 1] &= P[I_n \text{ is not erased } | I_n = 1]P[I_n = 1] \\
&= (1-\alpha)p \text{ where } I_n \text{ is Bernoulli process}
\end{aligned}$$

The Y_n are then a Bernoulli process with success probability

$$(1-\alpha)p \triangleq p'.$$

S'_n is then the binomial counting process with

$$P[S'_n = k] = \binom{n}{k} p'^k (1-p')^{n-k}$$

S'_n has independent and stationary increments.

6.31

$$P[N(15) - N(10) = 0, N(60) - N(45) = 0]$$
$$= P[N(15) - N(10) = 0]\ P[N(60) - N(45) = 0] = e^{-\lambda 15}\ e^{-\lambda 15} = e^{-5}\,,$$

since $\lambda = 10/60$.

6.34 Let $X_i =$ time till first arrival in line i

a)
$$\begin{aligned} P[X_1 < X_2] &= \int_0^\infty P[X_2 > x | X_1 = x] f_{X_1}(x) dx \\ &= \int_0^\infty e^{-\lambda_2 x} \lambda_1 e^{-\lambda_1 x} dx = \lambda_1 \int_0^\infty e^{-(\lambda_1+\lambda_2)x} dx \\ &= \frac{\lambda_1}{\lambda_1 + \lambda_2} \end{aligned}$$

b) Time till first arrival $\triangleq Z = \min(X_1, X_2)$

$$\begin{aligned} P[\min(X_1, X_2) > x\,] &= P[X_1 > x, X_2 > x] = P[X_1 > x]P[X_2 > x] \\ &= e^{-\lambda_1 x} e^{-\lambda_2 x} = e^{-(\lambda_1+\lambda_2)x} \end{aligned}$$

$\Rightarrow Z$ exponential RV with mean $(\lambda_1 + \lambda_2)^{-1}$

c)
$$\begin{aligned} G_{N(t)}(z) &= \mathcal{E}[z^{N_1(t)+N_2(t)}] = \mathcal{E}[z^{N_1(t)}]\mathcal{E}[z^{N_2(t)}] \\ &= e^{\lambda_1(z-1)} e^{\lambda_2(z-1)} = e^{(\lambda_1+\lambda_2)(z-1)} \end{aligned}$$

$\Rightarrow N(t)$ Poisson with rate $\lambda_1 + \lambda_2$

d) $G_{N(t)} = e^{(\lambda_1+\lambda_2+\ldots+\lambda_n)(z-1)}$ $\Rightarrow$ $N(t)$ is Poisson with rate $\lambda_1 + \cdots + \lambda_k$

6.35

$$P[N(t-d) = j \,|\, N(t) = k] =$$

$$= \frac{P[N(t-d) = j, N(t) = k]}{P[N(t) = k]}$$

$$= \frac{P[N(t-d) = j] P[N(t) - N(t-d) = k - j]}{P[N(t) = k]}$$

$$\frac{\frac{\lambda^j (t-d)^j}{j!} e^{-\lambda(t-d)} \frac{\lambda^{k-j} d^{k-j}}{(k-j)!} e^{-\lambda d}}{\frac{\lambda^k t^k}{k!} e^{-\lambda t}}$$

$$= \binom{k}{j} \left(\tfrac{t-d}{t} \right)^j \left(\tfrac{d}{t} \right)^{k-j} \quad \text{binomial}$$

6.40 $Y(t)$ is a random telegraph process with transition rate $p\alpha$, since if T = time till next transition, then

$$T = \tau_1 + \ldots + \tau_N \quad \text{where } N \text{ is geometric RV with prob. of success} = p,$$

and τ_i's are iid with exponential dist. with parameter α. We then have

$$\begin{aligned}
\phi_T(\omega) &= \mathcal{E}[\mathcal{E}[e^{j\omega T}|N]] = \mathcal{E}\left[\left(\frac{\alpha}{\alpha - j\omega}\right)^N\right] = \sum_{k=1}^{\infty}\left(\frac{\alpha}{\alpha - j\omega}\right)^k p(1-p)^{k-1} \\
&= \frac{\alpha p}{\alpha - j\omega}\frac{1}{1 - \frac{\alpha(1-p)}{\alpha - j\omega}} = \frac{\alpha p}{\alpha p - j\omega} \Rightarrow T \text{ has an exp. dist. with parameter } \alpha p.
\end{aligned}$$

$$\therefore P[Y(t) = +1] = \frac{1}{2} = P[Y(t) = -1] \text{ if } P[Y(0) = \pm 1] = \tfrac{1}{2}$$

and

$$C_Y(t_1, t_2) = e^{-2\alpha p|t_2 - t_1|}$$

6.48 a) we know that $Z(t) = X(t) - aX(t-s)$ is a Gaussian RV since $X(t)$ and $X(t-s)$ are jointly Gaussian. Therefore we need only find $m_Z(t)$ and $VAR[Z(t)]$

$$\begin{aligned}
m_Z(t) &= \mathcal{E}[X(t)] - a\mathcal{E}[X(t-s)] = 0 \\
VAR[Z(t)] &= \mathcal{E}[(X(t) - aX(t-s))^2] \\
&= \mathcal{E}[X^2(t)] - 2a\mathcal{E}[X(t)X(t-s)] + a^2\mathcal{E}[X^2(t-s)] \\
VAR[Z(t)] &= \lambda t - 2a(\lambda(t-s)) + a^2\lambda(t-s) \quad , t > s > 0 \\
&= \lambda t(1 - 2a + a^2) + 2a\lambda s - a^2\lambda s \\
&= \lambda t(a-1)^2 - a\lambda s(a-2) \\
f_{Z(t)}(z) &= \frac{\exp\left\{-\frac{z^2}{2VAR[Z(t)]}\right\}}{\sqrt{2\pi VAR[Z(t)]}}
\end{aligned}$$

b)

$$\begin{aligned}
m_Z(t) &= E[X(t) - aX(t-s)] = 0 \\
C_Z(t_1, t_2) &= E[\{Z(t_1) - m_Z(t_1)\}\{Z(t_2) - m_Z(t_2)\}] \\
&= E[\{X(t_1) - aX(t_1 - s)\}\{X(t_2) - aX(t_2 - s)\}] \\
&= \lambda \min(t_1, t_2) - a\lambda \min(t_1 - s, t_2) \\
&\quad - a\lambda \min(t_1, t_2 - s) + a^2\lambda[\min(t_1, t_2) - s]
\end{aligned}$$